The 33rd Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 22nd April 2026
Category II

Problem 1 *Let α be a real number. Evaluate the sum*

$$\sum_{n=1}^{\infty} 4^n \sin^4\left(2^{-n}\alpha\right).$$

[A. Yagmyrov / The State Energy Institute of Turkmenistan, Mary, Turkmenistan]

Solution Since $\sin^4 x = \sin^2 x - \frac{1}{4}\sin^2 2x$, we have

$$4^n \sin^4(2^{-n}\alpha) = 4^n \sin^2(2^{-n}\alpha) - 4^{n-1}\sin^2(2^{-(n-1)}\alpha).$$

So this is a telescoping series, with sum equal to

$$\begin{aligned}\sum_{n=1}^{\infty} 4^n \sin^4(2^{-n}\alpha) &= \lim_{n\to\infty} 4^n \sin^2(2^{-n}\alpha) - \sin^2\alpha \\ &= \lim_{n\to\infty} \left(\frac{\sin(2^{-n}\alpha)}{2^{-n}}\right)^2 - \sin^2\alpha \\ &= \alpha^2 - \sin^2\alpha\end{aligned}$$

□

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Problem 2 *Let n be an odd number and k be a positive integer. Find all real $n \times n$ matrices A such that*

$$\operatorname{tr}(A) = 0\,,$$
$$\operatorname{tr}(A^{k+1}) = n^2 - 1$$

and the sum

$$I + A + A^2 + \cdots + A^k$$

is the diagonal $n \times n$ matrix with entries $(n, 1, -1, 1, -1, \ldots, 1, -1)$ on the main diagonal.

Here $\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii}$ is the trace of a matrix X, and I is the identity matrix.

[Slobodan Filipovski / University of Primorska, Koper]

Solution For $n = 1$, it is easy to note that the zero matrix $A = [0]$ satisfies the conditions of the problem for every $k \in \mathbb{N}$. Therefore, we assume that $n \geq 3$.

Since D is a diagonal matrix, its spectrum is $\{n, 1^{\left(\frac{n-1}{2}\right)}, (-1)^{\left(\frac{n-1}{2}\right)}\}$. The spectrum of A consists of:

1. one eigenvalue d such that $1 + d + d^2 + \cdots + d^k = n$.
2. $\frac{n-1}{2}$ eigenvalues λ_i, $1 \leq i \leq \frac{n-1}{2}$, such that $1 + \lambda_i + \lambda_i^2 + \ldots + \lambda_i^k = 1$ for $1 \leq i \leq \frac{n-1}{2}$, and
3. $\frac{n-1}{2}$ eigenvalues μ_i, $1 \leq i \leq \frac{n-1}{2}$, such that $1 + \mu_i + \mu_i^2 + \ldots + \mu_i^k = -1$ for $1 \leq i \leq \frac{n-1}{2}$.

Each eigenvalue λ_i satisfies $1 + \lambda_i + \lambda_i^2 + \ldots + \lambda_i^k = 1$. Thus we get $\lambda_i(\lambda_i^k - 1) = 0$, which implies λ_i is either zero or a k-th root of unity. Hence

$$\sum_{i=1}^{\frac{n-1}{2}} \lambda_i^{k+1} = \sum_{i=1}^{\frac{n-1}{2}} \lambda_i. \tag{1}$$

Similarly each eigenvalue μ_i satisfies $\mu_i^{k+1} = 2 - \mu_i$, so that

$$\sum_{i=1}^{\frac{n-1}{2}} \mu_i^{k+1} = 2 \cdot \frac{n-1}{2} - \sum_{i=1}^{\frac{n-1}{2}} \mu_i = n - 1 - \sum_{i=1}^{\frac{n-1}{2}} \mu_i. \tag{2}$$

Moreover, from $\mu_i^{k+1} = 2 - \mu_i$ and from the triangle inequality we conclude that $|\mu_i| \leq 2$.

From $\operatorname{tr}(A) = 0$ we get

$$\sum_{i=1}^{\frac{n-1}{2}} \lambda_i + \sum_{i=1}^{\frac{n-1}{2}} \mu_i + d = 0, \tag{3}$$

and $d \in \mathbb{R}$ (since $\{d\}$, $\{\lambda_1, \ldots, \lambda_{\frac{n-1}{2}}\}$ and $\{\mu_1, \ldots, \mu_{\frac{n-1}{2}}\}$ are pairwise disjoint and closed sets with respect to complex conjugation).

Similarly, from $\operatorname{tr}(A^{k+1}) = n^2 - 1$ and by using (1) and (2) we get

$$n^2 - 1 = \sum_{i=1}^{\frac{n-1}{2}} \lambda_i^{k+1} + \sum_{i=1}^{\frac{n-1}{2}} \mu_i^{k+1} + d^{k+1} = \sum_{i=1}^{\frac{n-1}{2}} \lambda_i - \sum_{i=1}^{\frac{n-1}{2}} \mu_i + n - 1 + d^{k+1}. \tag{4}$$

Now, by subtracting (3) and (4) and by using the identity $d^{k+1} = nd - n + 1$ we get

$$\sum_{i=1}^{\frac{n-1}{2}} \mu_i = \frac{(n-1)(d-1-n)}{2}. \tag{5}$$

We prove that $n \geq d + 1$. If $d \leq 1$, then $n \geq 3 > d + 1$. If $d > 1$, from $nd - n + 1 = d^{k+1} \geq d^2$ we get $n \geq d + 1$. Now, based on (5) we get

$$\left|\frac{(n-1)(d-1-n)}{2}\right| = \frac{(n-1)(n+1-d)}{2} = \left|\sum_{i=1}^{\frac{n-1}{2}} \mu_i\right| \leq \sum_{i=1}^{\frac{n-1}{2}} |\mu_i| \leq n - 1. \tag{6}$$

The inequality in (6) implies $n \leq d+1$. Together with $n \geq d+1$, we have $n = d+1$.
If $k \geq 2$, then $n \geq 1 + d + d^2 > 1 + d$, which is not possible. Thus $k = 1$.

In this case ($k = 1$), we solve the equation $I + A = D$, which leads to $A = \mathrm{diag}(n-1, 0, -2, \ldots, 0, -2)$. It is easy to check that $\mathrm{tr}(A) = 0$ and $\mathrm{tr}(A^2) = n^2 - 1$. □

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Problem 3 *Let n be an integer greater than 2, and let x and y be positive integers such that*

$$x + y^n \quad \textit{is divisible by} \quad z = \frac{(x+y)^n - y^n}{x} - 1\,.$$

Prove that z is divisible by the n-th power of some integer greater than 1.

[Mirela Jukić Bokun / J.J. Strossmayer University of Osijek]

Solution Since $x + y^n = dz$, with d a positive integer, from the definition of z we have

$$(x+y)^n = xz + x + y^n = z(x+d),$$

i.e., z divides $(x+y)^n$. Let $x + y = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ be the prime factiorization of $x + y$. Then $z = p_1^{\beta_1} \cdots p_k^{\beta_k}$, $0 \le \beta_i \le n\alpha_i$, $i = 1, \ldots, k$. Suppose that z is not divisible by the n-th power of an integer greater than 1. Then $0 \le \beta_i \le n - 1 \le (n-1)\alpha_i$, $i = 1, \ldots, k$ and z divides $(x+y)^{n-1}$. From the definition of z and the binomial theorem, we obtain:

$$z = x^{n-1} + \binom{n}{1} x^{n-2}y + \cdots + \binom{n}{n-2} xy^{n-2} + \binom{n}{n-1} y^{n-1} - 1.$$

It follows

$$\begin{aligned}
(x+y)^{n-1} &= x^{n-1} + \binom{n-1}{1} x^{n-2}y + \cdots + \binom{n-1}{n-2} xy^{n-2} + y^{n-1} \\
&< x^{n-1} + \binom{n}{1} x^{n-2}y + \cdots + \binom{n}{n-2} xy^{n-2} + y^{n-1} \\
&= z + y^{n-1}(1-n) + 1 < z.
\end{aligned}$$

Since z is a positive integer greater than 1 that divides $(x+y)^{n-1}$, this is a contradiction. Thus, z is divisible by the n-th power of an integer greater than 1. □

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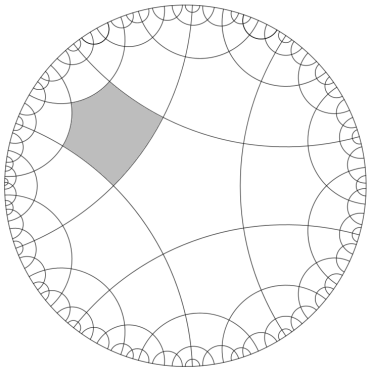
Problem 4 *The unit disk with center O has been tessellated with infinitely many non-overlapping tiles such that*

- *the tiles are bounded by circle arcs, perpendicular to the unit circle;*
- *the neighboring tiles are reflections (inverses) of each other in the common boundary arc.*



An example of a tessellation with regular pentagons, one tile is filled grey

Let $V_1, V_2, \ldots$ be the vertices in the tessellation, i.e. the points in the disk where three or more tiles meet. Prove that

$$\prod_{k=1}^{\infty} |OV_k| = 0\,.$$

[Géza Kós / Loránd Eötvös University, Budapest]

Solution Place the figure in the complex plane, so $O = 0$ and $V_1, V_2 \ldots$ are complex numbers. Consider the Blaschke product

$$f(z) = \prod_{k=1}^{\infty} \frac{|V_k|}{V_k} \frac{V_k - z}{1 - \overline{V_k} z};$$

It is well-kown that this product converges to a holomorphic function.

Note that

$$\big|f(0)\big| = \prod_{k=1}^{\infty} |V_k|.$$

This function has many symmetries: the composition of any two reflections (inversions) is a period. Hence, all values of f are attained in any two neighboring tiles, that is a compact set. So, the function is bounded. By Weierstrass' theorem, $|f|$ has a maximal value, attained inside the disk.

By the maximum modulus principle, the function $|f|$ cannot have any local maximum, except if f is constant. So, we conclude that f is constant. Since f has a root at every V_k, $f \equiv 0$. In particular, $f(0) = 0$. □