The 33rd Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 22nd April 2026
Category I

Problem 1 *Let n be an odd integer and let A and B be complex $n \times n$ matrices such that $A, B, A - B$ are invertible and*

$$(A - B)^{-1} = A^{-1} - B^{-1}\,.$$

Prove that

$$\det A \det B = -(\det(A - B))^2\,.$$

[Darija Brajković Zorić / J.J. Strossmayer University of Osijek]

Solution We can write $A^{-1} - B^{-1}$ as $A^{-1}(B - A)B^{-1}$ so we now have

$$(A - B)^{-1} = A^{-1}(B - A)B^{-1}.$$

Then $\det(A - B)^{-1} = \det(A^{-1}(B - A)B^{-1})$, so

$$\frac{1}{\det(A - B)} = \frac{1}{\det A} \det(B - A) \frac{1}{\det B}.$$

Since $\det(B - A) = (-1)^n \det(A - B)$, we now have

$$\frac{1}{\det(A - B)} = \frac{(-1)^n \det(A - B)}{\det A \det B},$$

and we at once obtain what we needed to prove. □

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Problem 2 *Géza has invited his $n \geq 2$ friends to dinner. At his round table there are exactly n seats placed equidistantly, and Géza placed name cards on the seats that completely determine the seating arrangement. Unfortunately, the friends ignored the name cards and just sat down arbitrarily.*

Depending on n, find the largest possible positive integer $k(n)$ with the following property: For any seating arrangement, Géza can rotate the table so that after the rotation at least $k(n)$ friends are seated correctly.

(A rotation by 0 positions is also considered a rotation.) [Robert Lauff / TU Berlin]

Solution It is clear that there is always a rotation in which at least one person is seated correctly.

Suppose n is odd. Consider the reverse permutation $\pi(i) = n - i + 1$. Define the distance vector

$$d(\pi)_i = (i - \pi(i)) \bmod n,$$

which indicates how far we must rotate the table counterclockwise so that person i sits correctly. Then

$$d(\pi)_i = (2i - 1) \bmod n,$$

so

$$d(\pi) = (1, 3, 5, \ldots, n-2, 0, 2, 4, \ldots, n-1).$$

Since all distances are distinct, exactly one person is seated correctly for every rotation.

Now suppose n is even. The distance vector of the reverse permutation is

$$d(\pi) = (1, 3, 5, \ldots, n-1, 1, 3, \ldots, n-1),$$

so there are at most 2 fixed points.

We now show that there are always at least 2 fixed points in the optimal rotation when n is even. Suppose, for contradiction, that there exists a permutation π such that all distances $d(\pi)_i$ are distinct. Then $d(\pi)$ is a permutation of the numbers $0, \ldots, n-1$, and thus

$$\sum_{i=1}^{n} d(\pi)_i = \sum_{k=0}^{n-1} k = \frac{(n-1)n}{2}.$$

On the other hand,

$$\sum_{i=1}^{n} d(\pi)_i \equiv \sum_{i=1}^{n} (i - \pi(i)) \equiv \sum_{i=1}^{n} i - \sum_{i=1}^{n} \pi(i) \equiv \frac{n(n+1)}{2} - \frac{n(n+1)}{2} \equiv 0 \pmod{n}.$$

Hence, $\sum_{i=1}^{n} d(\pi)_i$ must be divisible by n. However,

$$\frac{n}{2}(n-1)$$

is not divisible by n when n is even, which is a contradiction. Therefore, such a permutation cannot exist.

- If n is odd, at least 1 person is seated correctly.
- If n is even, at least 2 people are seated correctly.

□

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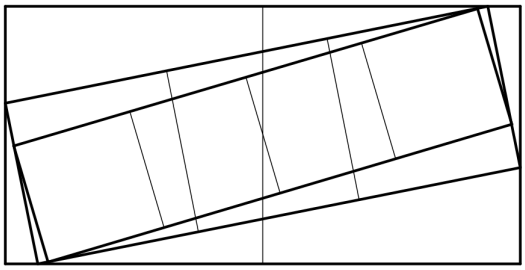
Problem 3 *Let us start with a rectangle R_2 consisting of two unit squares. Let us consider*

- *a rectangle R_3 – consisting of three identical squares in a row – inscribed in R_2 (this means that vertices of R_3 lie on distinct sides of R_2),*
- *a rectangle R_4 – consisting of four identical squares in a row – inscribed in R_3,*
- *...*
- *a rectangle R_{n+1} – consisting of $n+1$ identical squares in a row – inscribed in R_n, and so on.*

Is it possible to put all rectangles next to each other in a row on a finite desk?
In other words: Denoting by a_n the length of the shorter side of R_n, decide whether $\sum_{n=2}^{\infty} a_n$ converges.



[Tomáš Bárta / Charles University, Prague]

Solution The sum diverges.

Let us compute the relation between a_n and a_{n+1}. Let $ABCD$ be a rectangle with $AB = a_n$, $BC = na_n$ and $KLMN$ be a rectangle with $KN = a_{n+1}$, $KL = (n+1)a_{n+1}$ and $K \in AB$, $L \in BC$, $M \in CD$, $N \in DA$. Denote $x = AK$ and $y = AN$. By similarity of triangles we have $BL = (n+1)x$, $BK = (n+1)y$. Therefore, $x + (n+1)y = a_n$, $y + (n+1)x = na_n$ which implies $x = a_n \frac{n^2+n-1}{n(n+2)}$, $y = a_n \frac{1}{n(n+2)}$ and

$$a_{n+1} = \sqrt{x^2 + y^2} = \frac{a_n}{n(n+2)} \sqrt{n^4 + 2n^3 - n^2 - 2n + 2}.$$

We have

$$\frac{a_{n+1}}{a_n} = \sqrt{\frac{n^4 + 2n^3 - n^2 + o(n^2)}{n^2(n+2)^2}} = \sqrt{\frac{1 + \frac{2}{n} - \frac{1}{n^2} + o(\frac{1}{n^2})}{1 + 4\frac{1}{n} + 4\frac{1}{n^2}}}.$$

Since

$$\frac{1 + 2x - x^2 + o(x^2)}{1 + 4x + 4x^2} = 1 - 2x + 3x^2 + o(x^2) \geq 1 - 2x, \qquad x \to 0$$

we have $\frac{a_{n+1}}{a_n} \geq \sqrt{1 - \frac{2}{n}}$ for all n large enough, say $n \geq n_0$. Then for $n \geq n_0$

$$a_n = a_{n_0} \prod_{k=n_0}^{n-1} \frac{a_{k+1}}{a_k} \geq a_{n_0} \prod_{k=n_0}^{n-1} \sqrt{\frac{k-2}{k}} = a_{n_0} \sqrt{\frac{(n_0-2)(n_0-1)}{(n-1)(n-2)}} = \frac{a_{n_0}}{n} \sqrt{\frac{(n_0-2)(n_0-1)}{1 - \frac{3}{n} + \frac{2}{n^2}}} \geq \frac{c}{n}.$$

Hence, $\sum a_n$ diverges. □

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Problem 4 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Suppose that there exists a positive number a such that*

$$g(x) = f(x) + a \int_{x-1}^{x} f(t)\,\mathrm{d}t\,, \quad x \in \mathbb{R}\,,$$

is a constant function. Show that f is a constant function.

[Henry Ricardo / Westchester Area Math Circle, NY, USA]

Solution Without loss of generality, we may replace f by $-f$ if necessary to assume that f is bounded below on $\mathbb{R}$. Let $m = \inf\limits_{x\in\mathbb{R}} f(x) \in \mathbb{R}$. Define $g(x) := f(x) - m$. Thus $g(x) \geq 0$, g is continuous on $\mathbb{R}$, and $\inf\limits_{x\in\mathbb{R}} g(x) = 0$.

From the assumption of the problem, there exists a constant C such that

$$f(x) + a \int_{x-1}^{x} f(t)\,dt = C, \quad \forall x \in \mathbb{R}.$$

Substitute $f = g + m$:

$$g(x) + m + a \int_{x-1}^{x} (g(t) + m)\ dt = C.$$

Rearranging,

$$g(x) + a \int_{x-1}^{x} g(t)\,dt + m(1 + a) = C.$$

Let $A = C - m(1 + a)$,

Then A is a constant and

$$g(x) + a \int_{x-1}^{x} g(t)\,dt = A, \quad \forall x \in \mathbb{R}. \tag{1}$$

Since $g(x) \geq 0$ and $\displaystyle\int_{x-1}^{x} g(t)\,dt \geq 0$, we have $A \geq 0$.

From (1),

$$g(x) = A - a \int_{x-1}^{x} g(t)\,dt.$$

The right-hand side is differentiable, hence g is differentiable. Differentiate both sides of (1):

$$g'(x) + a[g(x) - g(x-1)] = 0. \tag{2}$$

Let $h(x) = e^{ax} g(x)$, differentiate and use equation (2):

$$h'(x) = e^{ax} g'(x) + ae^{ax} g(x) = ae^{ax} \left(g(x) + \frac{1}{a} g'(x) \right) = ae^{ax} g(x-1) \geq 0.$$

Therefore $h(x)$ is increasing on $\mathbb{R}$.

Since h is increasing, for $t \in [x-1, x]$, $h(t) \leq h(x)$, that is,

$$e^{at} g(t) \leq e^{ax} g(x) \implies g(t) \leq e^{a(x-t)} g(x).$$

Integrating over $t \in [x-1, x]$:

$$\int_{x-1}^{x} g(t)\,dt \leq g(x) \int_{x-1}^{x} e^{a(x-t)}\,dt.$$

Compute the integral:

$$\int_{x-1}^{x} e^{a(x-t)}\,dt = \int_{0}^{1} e^{au}\,du = \frac{e^a - 1}{a}.$$

Substitute into (1):

$$A = g(x) + a \int_{x-1}^{x} g(t)\,dt \leq g(x) + a \cdot \frac{e^a - 1}{a} g(x) = e^a g(x).$$

Hence

$$g(x) \geq Ae^{-a}, \quad \forall x \in \mathbb{R}. \tag{3}$$

From which taking infimum over x gives $\inf_{x\in\mathbb{R}} g(x) \geq \inf_{x\in\mathbb{R}} Ae^{-a} = Ae^{-a}$.

If $A > 0$, then $0 = \inf_{x\in\mathbb{R}} g(x) \geq Ae^{-a} > 0$ which is a contradiction. Therefore $A = 0$. Substituting $A = 0$ into (1):

$$g(x) + a\int_{x-1}^{x} g(t)\,dt = 0.$$

Since $g(x) \geq 0$ and $\int_{x-1}^{x} g(t)\,dt \geq 0$, $g(x) = 0$. Thus $f(x) = m$ is a constant function. $\square$