The 33rd Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 22nd April 2026
Category II

**Problem 1** *Let $\alpha$ be a real number. Evaluate the sum*

$$\sum_{n=1}^{\infty} 4^n \sin^4\left(2^{-n}\alpha\right).$$

[10 points]

**Problem 2** *Let $n$ be an odd number and $k$ be a positive integer. Find all real $n \times n$ matrices $A$ such that*

$$\operatorname{tr}(A) = 0,$$
$$\operatorname{tr}(A^{k+1}) = n^2 - 1$$

*and the sum*

$$I + A + A^2 + \cdots + A^k$$

*is the diagonal $n \times n$ matrix with entries $(n, 1, -1, 1, -1, \ldots, 1, -1)$ on the main diagonal.*

*Here $\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii}$ is the trace of a matrix $X$, and $I$ is the identity matrix.* [10 points]

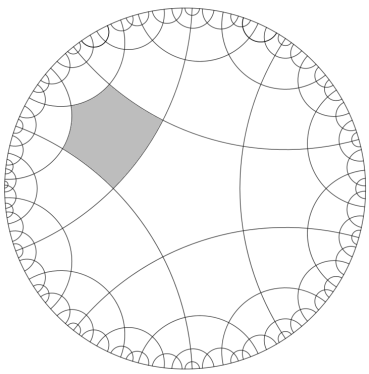
**Problem 3** *Let $n$ be an integer greater than 2, and let $x$ and $y$ be positive integers such that*

$$x + y^n \quad \text{is divisible by} \quad z = \frac{(x+y)^n - y^n}{x} - 1.$$

*Prove that $z$ is divisible by the $n$-th power of some integer greater than 1.* [10 points]

**Problem 4** *The unit disk with center $O$ has been tessellated with infinitely many non-overlapping tiles such that*

- *the tiles are bounded by circle arcs, perpendicular to the unit circle;*
- *the neighboring tiles are reflections (inverses) of each other in the common boundary arc.*



*An example of a tessellation with regular pentagons,*
*one tile is filled grey*

*Let $V_1, V_2, \ldots$ be the vertices in the tessellation, i.e. the points in the disk where three or more tiles meet. Prove that*

$$\prod_{k=1}^{\infty} |OV_k| = 0.$$

[10 points]