The 31$^{\text{st}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13$^{\text{th}}$ April 2024
Category I

Problem 1 *Let $f\colon \mathbb{R}\to\mathbb{R}$ be a continuously differentiable function. Prove that*

$$\left|f(1)-\int_0^1 f(x)\,\mathrm{d}x\right| \le \frac{1}{2}\max_{x\in[0,1]}\left|f'(x)\right|.$$

[Robert Skiba / Nicolaus Copernicus University in Toruń]

Solution We have

$$\begin{aligned}
f(1)-\int_0^1 f(x)dx &= \int_0^1 f'(x)dx-\int_0^1 f(x)dx\\
&= \int_0^1 f'(x)dx-\int_0^1\int_0^x f'(y)dydx\\
&= \int_0^1 f'(x)dx-\int_0^1\int_y^1 f'(y)dxdy\\
&= \int_0^1 f'(x)dx-\int_0^1 f'(y)\int_y^1 dxdy\\
&= \int_0^1 f'(x)dx-\int_0^1 f'(y)(1-y)dy\\
&= \int_0^1 f'(x)dx-\int_0^1 f'(x)(1-x)dx\\
&= \int_0^1 f'(x)(1-(1-x))dx\\
&= \int_0^1 f'(x)xdx.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\left|f(1)-\int_0^1 f(x)dx\right| &\le \left|\int_0^1 f'(x)xdx\right| \le \int_0^1 |f'(x)x|dx \le \int_0^1 \max_{x\in[0,1]}|f'(x)|\cdot|x|dx\\
&\le \max_{x\in[0,1]}|f'(x)|\int_0^1 |x|dx = \frac{1}{2}\max_{x\in[0,1]}|f'(x)|.
\end{aligned}$$

This completes the solution. $\square$

The 31$^{\text{st}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13$^{\text{th}}$ April 2024
Category I

Problem 2 *Let n be a positive integer and let A, B be two complex nonsingular $n \times n$ matrices such that*

$$A^2B - 2ABA + BA^2 = 0\,.$$

Prove that the matrix $AB^{-1}A^{-1}B - I_n$ is nilpotent. (Here I_n denotes the $n \times n$ identity matrix. A matrix X is called nilpotent if there exists a positive integer k such that $X^k = 0$.)

[Pasha Zusmanovich / University of Ostrava]

Solution It is enough to prove that 1 is the only eigenvalue of $AB^{-1}A^{-1}B$.

Lemma *If λ is an eigenvalue of $AB^{-1}A^{-1}B$, then $\frac{2\lambda-1}{\lambda}$ is an eigenvalue of $AB^{-1}A^{-1}B$.*

Proof Since $AB^{-1}A^{-1}B$ is nondegenerate, $\lambda \neq 0$, and $AB^{-1}A^{-1}B - \lambda E$ is degenerate. Then

$$BA^{-1}\Big(AB^{-1}A^{-1}B - \lambda E\Big) = \lambda A^{-1}(BA - AB)A^{-1} + (1-\lambda)A^{-1}B \tag{1}$$

is degenerate.

The condition $A^2B-2ABA+BA^2 = 0$ is equivalent to the condition that A commutes with $AB-BA$, hence A^{-1} commutes with $AB-BA$, and the right-hand side of (1) can be rewritten as $\lambda A^{-2}(BA-AB)+(1-\lambda)A^{-1}B$.

Hence

$$\frac{1}{\lambda}B^{-1}A\Big(\lambda A^{-2}(BA-AB) + (1-\lambda)A^{-1}B\Big) = B^{-1}A^{-1}BA - \frac{2\lambda-1}{\lambda}E$$

is degenerate, i.e., $\frac{2\lambda-1}{\lambda}$ is an eigenvalue of $B^{-1}A^{-1}BA$.

The matrices $B^{-1}A^{-1}BA$ and $AB^{-1}A^{-1}B$ are conjugate by A, hence they have the same eigenvalues, so $\frac{2\lambda-1}{\lambda}$ is also an eigenvalue of $AB^{-1}A^{-1}B$. □

Iterating the lemma, we get that for any eigenvalue λ of $AB^{-1}A^{-1}B$, and any integer $k \geq 1$,

$$\frac{k\lambda - (k-1)}{(k-1)\lambda - (k-2)}$$

is also an eigenvalue of $AB^{-1}A^{-1}B$. Since $AB^{-1}A^{-1}B$ has only a finite number of eigenvalues, we have

$$\frac{k\lambda - (k-1)}{(k-1)\lambda - (k-2)} = \frac{k'\lambda - (k'-1)}{(k'-1)\lambda - (k'-2)}$$

for some (actually, infinitely many) $k \neq k'$. The last equality is equivalent to $(k-k')(\lambda-1)^2 = 0$, whence $\lambda = 1$. □

The 31st Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13th April 2024
Category I

Problem 3 *Let n be a positive integer and let G be a simple undirected graph on n vertices. Let d_i be the degree of its i-th vertex, $i = 1, \dots, n$. Denote $\Delta = \max d_i$. Prove that if*

$$\sum_{i=1}^{n} d_i^2 > n\Delta(n - \Delta)$$

then G contains a triangle. (A graph is called simple if there are no loops and no multiple edges between any pair of vertices.) [Slobodan Filipovski / University of Primorska, Koper]

Solution We prove the claim by contraposition assuming that the obtained graph G does not contain triangles. If the i-th and the j-th vertex are connected we denote $i \sim j$. In this case holds $d_i + d_j \le n$. Hence

$$\sum_{i=1}^{n} d_i^2 = \sum_{i \sim j} (d_i + d_j) \le mn, \tag{1}$$

where m is the number of edges in the graph.
Let v be a vertex of G with maximum degree Δ. Since G is a triangle-free graph there are no edges in the neighbourhood of v. Moreover, every vertex which is not in the neighborhood of v has degree at most Δ. Therefore, the maximum number of edges of G is

$$m \le \Delta + (n - \Delta - 1)\Delta = \Delta(n - \Delta). \tag{2}$$

From (1) and (2) we get

$$\sum_{i=1}^{n} d_i^2 \le mn \le n\Delta(n - \Delta). \tag{3}$$

□

The 31$^{\text{st}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13$^{\text{th}}$ April 2024
Category I

Problem 4 *Let $p > 2$ be a prime and let*

$$\mathcal{A} = \left\{ n \in \mathbb{N} : 2p \mid n \text{ and } p^2 \nmid n \text{ and } n \mid 3^n - 1 \right\}.$$

Prove that

$$\limsup_{k\to\infty} \frac{\left|\mathcal{A} \cap [1,k]\right|}{k} \le \frac{2\log 3}{p \log p}.$$

[Slobodan Filipovski / University of Primorska, Koper]

Solution Let $n \in \mathcal{A}_p$. Then $p \mid (3^{\frac{n}{2}} - 1)(3^{\frac{n}{2}} + 1)$, from where $3^{\frac{n}{2}} \equiv 1 \pmod p$ or $3^{\frac{n}{2}} \equiv -1 \pmod p$. Since $p \mid n$ and n is an even number, $n = pr$, where r is even. Since $(p,3) = 1$, Fermat's little theorem yields $3^{\frac{n}{2}} \equiv (3^p)^{\frac{r}{2}} \equiv 3^{\frac{r}{2}} \pmod p$. Hence, $3^{\frac{r}{2}} \equiv 1 \pmod p$ or $3^{\frac{r}{2}} \equiv -1 \pmod p$. Recalling $(p,3) = 1$ again, let l denote the smallest positive integer satisfying $3^l \equiv 1 \pmod p$. This yields $p < 3^l$, and therefore $l > \dfrac{\log p}{\log 3}$. As shown above, there are two possible residue classes modulo l that $\dfrac{r}{2}$ might belong to. Thus, the asymptotic density of the multiples rp for which r satisfies the above conditions within the set of all multiples of p is at most $2 \cdot \dfrac{\log 3}{\log p}$. To determine the asymptotic density of the multiples of p within the set of all positive integers, we can consider the set $M_k = \{p, 2p, 3p, \ldots, mp\}$ with $mp \le k$, for a positive integer k. Then $|M_k| = m$, and therefore

$$\overline{d}(M_k) = \limsup_{k\to\infty} \frac{m}{k} \le \frac{1}{p}.$$

By these observations we get

$$\overline{d}(\mathcal{A}_p) = \limsup_{k\to\infty} \frac{|\mathcal{A}_p \cap [1,k]|}{k} < \frac{1}{p} \cdot \frac{2}{l} \le \frac{2\log 3}{p\log p}.$$

□