The 31st Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13th April 2024
Category II

Problem 1 *Suppose that* $f\colon[-1,1]\to\mathbb{R}$ *is continuous and that*

$$\left(\int_{-1}^{1} \mathrm{e}^x f(x)\,\mathrm{d}x\right)^2 \geq \left(\int_{-1}^{1} f(x)\,\mathrm{d}x\right)\left(\int_{-1}^{1} \mathrm{e}^{2x} f(x)\,\mathrm{d}x\right).$$

Prove that there exists a point $c\in(-1,1)$ *such that* $f(c)=0$. [10 points]

Problem 2 *A real* 2024×2024 *matrix* A *is called nice if* $(Av,v)=1$ *for every vector* $v\in\mathbb{R}^{2024}$ *with unit norm.*
a) Prove that the only nice matrix such that all of its eigenvalues are real is the identity matrix.
b) Find an example of a nice non-identity matrix. [10 points]

Problem 3 *Let* $a_1>0$ *and for* $n\geq 1$ *define*

$$a_{n+1}=a_n+\frac{1}{a_1+a_2+\ldots+a_n}.$$

Prove that $\lim\limits_{n\to\infty}\dfrac{a_n^2}{\ln n}=2$. [10 points]

Problem 4 *Let* $(b_n)_{n\geq 0}$ *be a sequence of positive integers satisfying* $b_n=d\left(\sum_{k=0}^{n-1} b_k\right)$ *for all* $n\geq 1$. *(By* $d(m)$ *we denote the number of positive divisors of* m.*)*
a) Prove that $(b_n)_{n\geq 0}$ *is unbounded.*
b) Prove that there are infinitely many n *such that* $b_n>b_{n+1}$. [10 points]