The 29th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 29th March 2019
Category I

Problem 1 *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence given recursively by* $a_0 = 1$ *and*

$$a_{n+1} = \frac{7a_n + \sqrt{45a_n^2 - 36}}{2}, \qquad n = 0, 1, \ldots$$

Show that the following statements hold for all positive integers n*:*

a) a_n *is a positive integer.*

b) $a_n a_{n+1} - 1$ *is the square of an integer.*

[Štefan Gyürki / Matej Bel University, Banská Bystrica]

Solution

(a) By simple counting $a_1 = 5$. Since $2a_{n+1} - 7a_n = \sqrt{45a_n^2 - 36} \geq 0$, the sequence $\{a_n\}$ is strictly increasing. From the last equation we have:

$$a_{n+1}^2 - 7a_n a_{n+1} + a_n^2 + 9 = 0 \qquad (1)$$
$$a_n^2 - 7a_{n-1}a_n + a_{n-1}^2 + 9 = 0.$$

Thus, by taking the difference of these equations we get

$$a_{n+1} = 7a_n - a_{n-1}$$

proving that a_n is always a positive integer.

(b) From (1) we get $(a_{n+1} + a_n)^2 = 9(a_n a_{n+1} - 1)$, hence

$$a_n a_{n+1} - 1 = \left(\frac{a_n + a_{n+1}}{3} \right)^2.$$

We already know that a_n and a_{n+1} are positive integers, therefore $(a_n + a_{n+1})/3$ is rational and its square is $a_n a_{n+1} - 1$ is a positive integer. Thus, $(a_n + a_{n+1})/3$ is also a positive integer and $a_n a_{n+1} - 1$ is a square.

□

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Problem 2 *A triplet of polynomials $u, v, w \in \mathbb{R}[x, y, z]$ is called* smart *if there exist polynomials $P, Q, R \in \mathbb{R}[x, y, z]$ such that the following polynomial identity holds:*

$$u^{2019}P + v^{2019}Q + w^{2019}R = 2019\,.$$

a) *Is the triplet of polynomials*

$$u = x + 2y + 3\,, \quad v = y + z + 2\,, \quad w = x + y + z$$

smart?

b) *Is the triplet of polynomials*

$$u = x + 2y + 3\,, \quad v = y + z + 2\,, \quad w = x + y - z$$

smart?

[Artūras Dubickas / Vilnius University]

Solution The answer is: a) no; b) yes.

In case a) the polynomials u, v, w are all equal to zero at $x = 2$, $y = -5/2$ and $z = 1/2$. This leads to $0 = 2019$, which is absurd.

In case b) we have $u - v - w = 1$. Consider the nth power of this identity with some $n \geq 3 \cdot 2018 + 1$:

$$(u - v - w)^n = \sum_{n_1+n_2+n_3=n} \frac{n!(-1)^{n_2+n_3}}{n_1!n_2!n_3!} u^{n_1} v^{n_2} w^{n_3} = 1.$$

By the choice of n, in each term of the sum at least one power n_i is greater than or equal to 2019. (It can be several n_i with this property.) We can thus rewrite the above identity in the form

$$u^{2019}P + v^{2019}Q + w^{2019}R = 1$$

with some $P, Q, R \in \mathbb{Z}[x, y, z]$. (The choice of P, Q, R is not unique.) Multiplying it by 2019 we conclude that the triplet (u, v, w) is useful. $\square$

Problem 3 *For an invertible $n \times n$ matrix M with integer entries we define a sequence $\mathcal{S}_M = \{M_i\}_{i=0}^{\infty}$ by the recurrence*

$$M_0 = M$$
$$M_{i+1} = (M_i^T)^{-1} M_i\,, \qquad i = 0, 1, \dots$$

Find the smallest integer $n \geq 2$ for which there exists a normal $n \times n$ matrix M with integer entries such that its sequence $\mathcal{S}_M$ is non-constant and has period $P = 7$, i.e., $M_{i+7} = M_i$ for all $i = 0, 1, \dots$

(M^T means the transpose of a matrix M. A square matrix M is called normal *if $M^T M = M M^T$ holds.)*

[Martin Niepel / Comenius University, Bratislava]

Solution For a normal square matrix M, we have its Schur decomposition $M = UDU^*$, where U is unitary and D is diagonal with eigenvalues of M on the diagonal. It follows, that if $M_i = UD_iU^*$ then $M_{i+1} = U\left(D_i^*\right)^{-1} D_i U^*$. So we can pass to eigenvalues and investigate periodic behavior of complex sequences given by the recurrence $\lambda_{i+1} = \lambda_i / \bar{\lambda}_i$. Consequently,

$$\lambda_n = \frac{\lambda_0^{2^{n-1}}}{\bar{\lambda}_0^{2^{n-1}}}.$$

Clearly, $|\lambda_i| = 1$ for $i \geq 1$, and if we require $\lambda_P = \lambda_0$, we arrive at the equation $\lambda_0 = \lambda_0^{2^P}$, so every eigenvalue λ_0 of M has to be $(2^P - 1)^{\text{st}}$ root of unity.

Since all eigenvalues of M satisfy $|\lambda| = 1$ and M is normal, M has to be unitary. The matrix M satisfies $M^{2^P-1} = I$, as well. Matrix M can not have all eigenvalues equal to 1, it would imply $M = I$, and the period would be 1 in that case. Hence, it has at least one primitive $(2^P - 1)^{\text{st}}$ root of unity as an eigenvalue and $2^P - 1$ is its multiplicative order.

The only unitary matrices with integer coeficients are signed permutiation matrices (i.e. matrices with exactly one ± 1 in every row and colummn, all other entries being zero). The order of a $n \times n$ signed permutatation matrix equals the order of the underlying permutation matrix or its double, and therefore has to be a divisor of $2n!$ – double of the order of a symmetric group S_n. Since $2^P - 1$ is a prime, it follows that $n \geq 2^P - 1$.

For $n = 2^P - 1$ such matrices, indeed, do exist, take a matrix of an action of a cyclic permutation on $\mathbb{R}^{2^P-1}$ given by $e_i \mapsto e_{i+1}$ for $i = 1, 2, \dots, 2^P - 2$ and $e_{2^P-1} \mapsto e_1$. $\square$

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Problem 4 *Determine the largest constant $K \geq 0$ such that*

$$\frac{a^a(b^2+c^2)}{(a^a-1)^2}+\frac{b^b(c^2+a^2)}{(b^b-1)^2}+\frac{c^c(a^2+b^2)}{(c^c-1)^2} \geq K\left(\frac{a+b+c}{abc-1}\right)^2$$

holds for all positive real numbers a, b, c such that $ab+bc+ca=abc$.

[Orif Ibrogimov / Czech Technical University of Prague]

Solution We show that the answer is $K = 18$ and that the equality holds if and only if $a = b = c = 3$. In view of the hypothesis of the problem, we deduce from

$$\left(\frac{1}{a}-\frac{1}{b}\right)^2+\left(\frac{1}{b}-\frac{1}{c}\right)^2+\left(\frac{1}{c}-\frac{1}{a}\right)^2 \geq 0,$$

that $abc \geq 3(a+b+c)$. This and the elementary inequalities $a^2+b^2 \geq 2ab$, $b^2+c^2 \geq 2bc$, $c^2+a^2 \geq 2ca$ complete the proof if we show that

$$\frac{a^{a-1}}{(a^a-1)^2}+\frac{b^{b-1}}{(b^b-1)^2}+\frac{c^{c-1}}{(c^c-1)^2} \geq \frac{abc}{(abc-1)^2}.$$

For $x \in (0,1)$, by differentiating both sides of the identity $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, we get the well-known identity

$$\frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k.$$

We use this identity for $x \in \{a^{-a}, b^{-b}, c^{-c}\}$. Since $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 1$ by the hypothesis, using Young's inequality, we thus obtain

$$\sum \frac{a^{a-1}}{(a^a-1)^2} = \sum \frac{1}{a}\frac{a^{-a}}{(1-a^{-a})^2} = \sum_{k=1}^{\infty} k\left(\frac{a^{-ak}}{a}+\frac{b^{-bk}}{b}+\frac{c^{-ck}}{c}\right) \geq \sum_{k=1}^{\infty} ka^{-k}b^{-k}c^{-k} = \frac{abc}{(abc-1)^2}.$$

It is easy to see that the equality holds if and only if $a = b = c = 3$. $\square$