The 29th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 29th March 2019
Category I

Problem 1 *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence given recursively by $a_0 = 1$ and*

$$a_{n+1} = \frac{7a_n + \sqrt{45a_n^2 - 36}}{2}, \qquad n = 0, 1, \ldots$$

Show that the following statements hold for all positive integers n:

a) *a_n is a positive integer.*

b) *$a_n a_{n+1} - 1$ is the square of an integer.*

[10 points]

Problem 2 *A triplet of polynomials $u, v, w \in \mathbb{R}[x, y, z]$ is called* smart *if there exist polynomials $P, Q, R \in \mathbb{R}[x, y, z]$ such that the following polynomial identity holds:*

$$u^{2019}P + v^{2019}Q + w^{2019}R = 2019\,.$$

a) *Is the triplet of polynomials*

$$u = x + 2y + 3\,, \quad v = y + z + 2\,, \quad w = x + y + z$$

smart?

b) *Is the triplet of polynomials*

$$u = x + 2y + 3\,, \quad v = y + z + 2\,, \quad w = x + y - z$$

smart?

[10 points]

Problem 3 *For an invertible $n \times n$ matrix M with integer entries we define a sequence $\mathcal{S}_M = \{M_i\}_{i=0}^{\infty}$ by the recurrence*

$$M_0 = M$$
$$M_{i+1} = (M_i^T)^{-1} M_i\,, \qquad i = 0, 1, \ldots$$

Find the smallest integer $n \geq 2$ for which there exists a normal $n \times n$ matrix M with integer entries such that its sequence $\mathcal{S}_M$ is non-constant and has period $P = 7$, i.e., $M_{i+7} = M_i$ for all $i = 0, 1, \ldots$

(M^T means the transpose of a matrix M. A square matrix M is called normal *if $M^T M = M M^T$ holds.)*

[10 points]

Problem 4 *Determine the largest constant $K \geq 0$ such that*

$$\frac{a^a(b^2 + c^2)}{(a^a - 1)^2} + \frac{b^b(c^2 + a^2)}{(b^b - 1)^2} + \frac{c^c(a^2 + b^2)}{(c^c - 1)^2} \geq K\left(\frac{a + b + c}{abc - 1}\right)^2$$

holds for all positive real numbers a, b, c such that $ab + bc + ca = abc$.

[10 points]