The 26th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 8th April 2016
Category I

Problem 1 *Let $f\colon \mathbb{R} \to (0,\infty)$ be a continuously differentiable function. Prove that there exists $\xi \in (0,1)$ such that*

$$\mathrm{e}^{f'(\xi)} f(0)^{f(\xi)} = f(1)^{f(\xi)}\,.$$

[10 points]

Solution The equality is equivalent to

$$\mathrm{e}^{f'(\xi)} = \left(\frac{f(1)}{f(0)}\right)^{f(\xi)}$$

and

$$\frac{f'(\xi)}{f(\xi)} = \ln f(1) - \ln f(0).$$

The existence of $\xi \in (0,1)$ satisfying the last equality follows from Lagrange's mean value theorem with the function $\phi(x) = \ln f(x)$, $x \in [0,1]$. $\square$

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Problem 2 *Find all positive integers n such that $\varphi(n)$ divides n^2+3.*
($\varphi(n)$ denotes Euler's totient function, i.e. the number of positive integers $k \le n$ coprime to n.) [10 points]

Solution Answer: $n = 1, 2, 3, 5, 9, 21$,

If n is prime, then $\varphi(n) = n-1$ divides $n^2+3 = (n-1)(n+1)+4$, thus $n-1$ divides 4 and we get answers $n = 2, 3, 5$. Let now $n > 1$ be composite and $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ has k distinct prime factors. Then $\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1)$ is divisible by 2^k, thus n^2+3 is divisible by 2^k. It implies that n is odd and $k \le 2$, since 8 never divides n^2+3. Next, if $\alpha_i \ge 2$, then p_i divides $\varphi(n)$, thus p_i divides n^2+3 and $p_i = 3$. In this case 9 does not divide n^2+3, hence 9 does not divide $\varphi(n)$, i.e. $\alpha_i = 2$. So, we have three cases: $n = 9$ (this is another answer), $n = 9p$ for prime $p \ne 3$ and $n = pq$ for different odd primes p, q.

1) $n = 9p$. Then $\varphi(n) = 6(p-1)$, $n^2+3 = 81(p-1)(p+1)+84$. So, $3(p-1)$ divides 84, thus $p-1$ divides 28, but for possible $p = 5, 29$ we again get that 8 divides $\varphi(n)$.

2) $n = pq$. If $q = 3$ we see that $\varphi(n) = 2(p-1)$ divides $n^2+3 = 9(p-1)(p+1)+12$, so $2(p-1)$ divides 12, for $p = 7$ we get the answer $n = 21$. Now assume that $p > 3$, $q > 3$. Then 3 does not divide n^2+3, thus 3 does not divide $\varphi(n) = (p-1)(q-1)$. It implies that both p and q are congruent to 2 modulo 3, and we may write $p = 2a+1$, $q = 2b+1$, where a, b are congruent to 2 modulo 3. We get that $\varphi(n) = 4ab$ divides $n^2+3 = (4ab+2a+2b+1)^2+3 = 4ab(4ab+1)+4(a^2+b^2+a+b+1)$, i.e. ab divides $a^2+b^2+a+b+1$. Let us prove that it is impossible when a, b are congruent to 2 modulo 3. Assume the contrary and choose a pair of such a, b with minimal value of $a+b$. Obviously a, b are coprime, so we may suppose that $a > b$ (here we use that $a = b = 1$ is forbidden modulo 3). The number b^2+b+1 is divisible by a, denote $b^2+b+1 = ma$. The number m is congruent to 2 modulo 3 and $m < a$ since $ma = b^2+b+1 < (b+1)^2 \le a^2$. Note that m divides b^2+b+1 and b divides $a^2(m^2+m+1) = (b^2+b+1)^2 + a(b^2+b+1) + a^2 = (a^2+a+1)+bx$ for some integer x. Since a and b are relatively prime, we get that b divides m^2+m+1. Thus both m and b divide $Q := b^2+m^2+b+m+1$, therefore bm divides Q and (b, m) is smaller pair then (a, b). This contradiction finishes the proof. $\square$

Problem 3 *Let $d \geq 3$ and let $A_1 \dots A_{d+1}$ be a simplex in $\mathbb{R}^d$. (A simplex is the convex hull of $d+1$ points not lying in a common hyperplane.) For every $i = 1, \dots, d+1$ let O_i be the circumcentre of the face $A_1 \dots A_{i-1}A_{i+1} \dots A_{d+1}$, i.e. O_i lies in the hyperplane $A_1 \dots A_{i-1}A_{i+1} \dots A_{d+1}$ and it has the same distance from all points $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{d+1}$. For each i draw a line through A_i perpendicular to the hyperplane $O_1 \dots O_{i-1}O_{i+1} \dots O_{d+1}$. Prove that either these lines are parallel or they have a common point.* [10 points]

Solution If $O_1, \dots, O_{d+1}$ lie in the hyperplane, our lines are parallel. If not, they form a simplex which has a circumcentre Q. Let O be circumcentre of $A_1 \dots A_{d+1}$, let P be symmetric to O in a point Q. Let's prove that all our lines pass through P. That is, PA_i must be perpendicular to O_jO_k if i, j, k are different. Denote by M_i a midpoint OA_i. Then A_iP is parallel to a middle line M_iQ of triangle OA_iP_i. We have $M_iO_j = M_iO_k = OA_i/2$, thus both points M_i, Q lie in a perpendicular bisector to the segment O_jO_k, hence $M_iQ \perp O_jO_k$ as desired. □

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Problem 4 *Find the value of the sum* $\sum_{n=1}^{\infty} A_n$, *where*

$$A_n = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{1}{k_1^2} \, \frac{1}{k_1^2 + k_2^2} \cdots \frac{1}{k_1^2 + \cdots + k_n^2} \, .$$

[10 points]

Solution We will show more general fact:

Theorem *Let* (a_n) *be an increasing sequence of real numbers greater or equal than 1, such that then series* $\sum_{n=1}^{\infty} \frac{1}{a_n}$ *converges to* S. *Then*

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{a_{n_1}} \, \frac{1}{a_{n_1} + a_{n_2}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_k}} = \frac{1}{k!} S^k$$

for every $k = 1, 2, \ldots$

Proof [I.] by induction on k

For $k = 1$ the equality $\sum_n \frac{1}{a_n} = S$ is obvious. Assume now that the equality holds for $k = 1, 2, \ldots, m-1$ and denote

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{a_{n_1}} \, \frac{1}{a_{n_1} + a_{n_2}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_k}} = \mathcal{S}_k \, .$$

We have

$$\frac{1}{a_{n_1}} \, \frac{1}{a_{n_1} + a_{n_2}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}} = \frac{(a_{n_1} + a_{n_2}) - a_{n_1}}{a_{n_2}} \, \frac{1}{a_{n_1}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}}$$

$$= \frac{1}{a_{n_2}} \, \frac{1}{a_{n_1}} \, \frac{1}{a_{n_1} + a_{n_2} + a_{n_3}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}} - \frac{1}{a_{n_2}} \, \frac{1}{a_{n_1} + a_{n_2}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}} .$$

The last term sums to $\mathcal{S}_m$ and the first equals to

$$\frac{1}{a_{n_2}} \, \frac{1}{a_{n_1}} \, \frac{1}{a_{n_1} + a_{n_2} + a_{n_3}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}} = \frac{(a_{n_1} + a_{n_2} + a_{n_3}) - a_{n_1}}{a_{n_2} + a_{n_3}} \, \frac{1}{a_{n_2}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}}$$

$$= \frac{1}{a_{n_2}} \, \frac{1}{a_{n_2} + a_{n_3}} \, \frac{1}{a_{n_1}} \, \frac{1}{a_{n_1} + a_{n_2} + a_{n_3} + a_{n_4}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}} - \frac{1}{a_{n_2}} \, \frac{1}{a_{n_2} + a_{n_3}} \, \frac{1}{a_{n_1} + a_{n_2} + a_{n_3}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}} .$$

Once again the last term sums to $\mathcal{S}_m$. Repeating the above transformations leads to

$$\frac{1}{a_{n_1}} \, \frac{1}{a_{n_1} + a_{n_2}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}} = \frac{1}{a_{n_2}} \, \frac{1}{a_{n_2} + a_{n_3}} \cdots \frac{1}{a_{n_2} + \cdots + a_{n_m}} \, \frac{1}{a_{n_1}} -$$

$$- \frac{1}{a_{n_2}} \, \frac{1}{a_{n_2} + a_{n_3}} \cdots \frac{1}{a_{n_2} + \cdots + a_{n_m}} \, \frac{1}{a_{n_1} + a_{n_2} + \cdots + a_{n_m}} - \cdots - \frac{1}{a_{n_2}} \, \frac{1}{a_{n_1} + a_{n_2}} \cdots \frac{1}{a_{n_1} + \cdots + a_{n_m}} .$$

So adding the above equality gives

$$\mathcal{S}_m = \mathcal{S}_{m-1} \cdot \mathcal{S}_1 - (m-1)\mathcal{S}_m \, ,$$

hence

$$\mathcal{S}_m = \frac{1}{m} \, \mathcal{S}_{m-1} \, \mathcal{S}_1 = \frac{1}{m} \, \frac{1}{(m-1)!} S^{m-1} \, S = \frac{1}{m!} S^m \, .$$

□

Proof [II.] in the case $a_n \geq n$

Let $F(x) = \sum_{n=1}^{\infty} x^{a_n}$. As $a_n \geq n$, the function F is well-defined and continuous for $x \in (-1, 1)$. We have $\int_0^1 F(x) \frac{dx}{x} = \sum_{n=1}^{\infty} \frac{1}{a_n} - 0 = S$. Moreover

$$F(x_1 x_2 \cdots x_k)\, F(x_2 \cdots x_k) \cdots F(x_k) \;=\; \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} x_1^{a_{n_1}}\, x_2^{a_{n_1}+a_{n_2}} \cdots x_k^{a_{n_1}+\cdots+a_{n_k}} \,,$$

hence

$$\sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{a_{n_1}}\, \frac{1}{a_{n_1}+a_{n_2}} \cdots \frac{1}{a_{n_1}+\cdots+a_{n_k}} \;=\; \int_0^1 \cdots \int_0^1 F(x_1 x_2 \cdots x_k)\, F(x_2 \cdots x_k) \cdots F(x_k)\, \frac{dx_1}{x_1}\, \frac{dx_2}{x_2} \cdots \frac{dx_k}{x_k} \,.$$

Let now $S(x) = \int_0^x F(t)\, \frac{dt}{t}$. We have of course $\frac{d}{dx}\, S(\alpha x)^n \;=\; n\, S(\alpha x)^{n-1}\, \frac{F(\alpha x)}{x}$. So

$$\begin{aligned}
&\int_0^1 \cdots \int_0^1 F(x_1 x_2 \cdots x_k)\, F(x_2 \cdots x_k) \cdots F(x_k)\, \frac{dx_1}{x_1}\, \frac{dx_2}{x_2} \cdots \frac{dx_k}{x_k} \\
&= \int_0^1 \Big(\cdots (\int_0^1 F(x_1 x_2 \cdots x_k)\, \frac{dx_1}{x_1}) \cdots F(x_k) \Big)\, \frac{dx_k}{x_k} \\
&= \int_0^1 \Big(\cdots (\int_0^1 S(x_1 x_2 \cdots x_k)\Big|_{x_1=0}^{x_1=1} F(x_2 \cdots x_k)\, \frac{dx_2}{x_2}) \cdots F(x_k) \Big)\, \frac{dx_k}{x_k} \\
&= \ldots = \frac{1}{k!} S^k \,.
\end{aligned}$$

□

Corollary *In the notation of the theorem we have* $\sum_{n=1}^{\infty} A_n = e^S - 1$, *where*

$$A_k = \sum_{n_1=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} \frac{1}{a_{n_1}}\, \frac{1}{a_{n_1}+a_{n_2}} \cdots \frac{1}{a_{n_1}+\cdots+a_{n_k}} \,.$$

Solution: It is known that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. From the Corollary we get directly that the sum in question is equal to $e^{\pi^2/6} - 1$. □