The 24th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 4th April 2014
Category I

Problem 1 *Find all complex numbers z such that $|z^3+2-2i|+z\bar{z}|z| = 2\sqrt{2}$. ($\bar{z}$ is the conjugate of z.)*

Solution

$$\sqrt{(-2)^2+2^2} = 2\sqrt{2} = |z^3+2-2i|+z\bar{z}|z| = |z^3-(-2+2i)|+|z^3|\,.$$

By the triangle inequality number z^3 must be a point of the straight line segment with ends 0 and $-2+2i = (1+i)^3$, so z must be a point of the union of the three straight line segments with the common end 0 and the remaining end equal to either $1+i$ or

$$(1+i)\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = \frac{1}{2}\left(-1-\sqrt{3}+i(\sqrt{3}-1)\right)$$

or

$$(1+i)\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = \frac{1}{2}\left(-1+\sqrt{3}-i(\sqrt{3}+1)\right)\,.$$

□

Second solution

Since $z\bar{z} = |z|^2 = |z^2|$ the equation may be rewritten as $|z^3+2-2i|+|z^3| = 2\sqrt{2}$. Let $z^3 = x+yi$ where $x, y \in \mathbb{R}$. The equation is equivalent to

$$\sqrt{(x+2)^2+(y-2)^2}+\sqrt{x^2+y^2} = 2\sqrt{2}. \tag{1}$$

Therefore

$$(x+2)^2+(y-2)^2 = 8-4\sqrt{2}\sqrt{x^2+y^2}+x^2+y^2\,,$$

so $x-y = \sqrt{2}\sqrt{x^2+y^2}$ and $(x+y)^2 = 0$, i.e. $y = -x$. Therefore the equation 1 takes the form

$$\sqrt{(x+2)^2+(-x-2)^2}+\sqrt{x^2+(-x)^2} = 2\,\sqrt{2} \tag{2}$$

which is equivalent to

$$|x+2|+|x| = 2. \tag{3}$$

Therefore $-2 \le x \le 0$. This means that $z^3 = x - xi$ for some $x \in [-2,0]$, i.e. $z^3 = r(\cos 135^\circ + i\sin 135^\circ)$ for some $r \in \left[0, 2\sqrt{2}\right]$. Therefore $z = \varrho\left(\cos(45^\circ + n\cdot 120^\circ) + i\sin(45^\circ + n\cdot 120^\circ)\right)$ with $n \in \{0,1,2\}$ and $0 \le \varrho \le \sqrt{2}$.

□

Problem 2 *We have a deck of $2n$ cards. Each shuffling changes the order from $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ to $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$. Determine all even numbers $2n$ such that after shuffling the deck 8 times the original order is restored.*

Solution Note that the cards a_1 and b_n always stay on the top/bottom of the deck respectively. From now on we will ignore the card b_n. Let us number the positions of the cards: $f(a_i) = i - 1$, $f(b_i) = n + i - 1$. Note that the shuffle will put the card with position i to position $2i$ for every $i < n$, or to $2i - (2n - 1)$ for every $n \leq i$.

This shows that the shuffling works like the mapping

$$\varphi\colon \mathbb{Z}_{2n-1} \to \mathbb{Z}_{2n-1},\ k \mapsto 2k.$$

Shuffling 8 times will map each k to $256k$. So we can reformulate the question:

For what numbers $2n$ will the following congruence hold for every $k \in \mathbb{Z}_{2n-1}$:

$$k \equiv 256k \pmod{2n-1}$$

It is easy to see that this congruence holds iff it is true for $k = 1$:

$$1 \equiv 256 \pmod{2n-1}$$

Which holds iff $2n - 1 \mid 255$. So the set we are looking for is: $\{2, 4, 6, 16, 18, 52, 86, 256\}$.

$\square$

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Problem 3 *Let* $n \geq 2$ *be an integer and let* $x > 0$ *be a real number. Prove that*

$$\left(1 - \sqrt{\tanh x}\right)^n + \sqrt{\tanh (nx)} < 1\,.$$

Recall that $\tanh t = \dfrac{e^{2t} - 1}{e^{2t} + 1}$.

Solution We will prove that for all real numbers x, $y > 0$

$$\left(1 - \sqrt{\tanh x}\right)\left(1 - \sqrt{\tanh y}\right) < 1 - \sqrt{\tanh(x+y)}. \tag{1}$$

Since

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \qquad \text{and} \qquad \tanh(x+y) = \frac{\sinh x \cosh y + \sinh y \cosh x}{\cosh x \cosh y + \sinh x \sinh y} = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

the inequality (1) is equivalent to

$$(1-u)\,(1-v) < 1 - \sqrt{\frac{u^2+v^2}{1+u^2v^2}} \qquad \text{for } 0 < u, v < 1 \tag{2}$$

via the substitutions $u := \sqrt{\tanh x}$ and $v := \sqrt{\tanh y}$. The inequality (2) can be shown as follows:

$$\begin{aligned}
&(1-u)(1-v) > 0 \\
\Longrightarrow\ & 2(1+uv) > (1+u)(1+v) \\
\Longrightarrow\ & 2uv > \frac{(1+u)(1+v)uv}{1+uv} = (1+u)(1+v) - \frac{(1+u)(1+v)}{1+uv} \\
\Longrightarrow\ & \frac{(1+u)(1+v)}{1+u^2v^2} > \frac{(1+u)(1+v)}{1+uv} > (1+u)(1+v) - 2uv = 2 - (1-u)(1-v) \\
\Longrightarrow\ & \frac{(1-u^2)(1-v^2)}{1+u^2v^2} > 2(1-u)(1-v) - (1-u)^2(1-v)^2 = 1 - (1-(1-u)(1-v))^2 \\
\Longrightarrow\ & \sqrt{\frac{u^2+v^2}{1+u^2v^2}} = \sqrt{1 - \frac{(1-u^2)(1-v^2)}{1+u^2v^2}} < |1-(1-u)(1-v)| = 1 - (1-u)(1-v) \\
\Longrightarrow\ & (1-u)(1-v) < 1 - \sqrt{\frac{u^2+v^2}{1+u^2v^2}}.
\end{aligned}$$

Thus we have shown (1) and from this the assertion follows by induction: Take $y = x$ for the case $n = 2$ and $y = nx$ for the inductive step $n \to n+1$. □

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Problem 4 *Let P_1, P_2, P_3, P_4 be the graphs of four quadratic polynomials drawn in the coordinate plane. Suppose that P_1 is tangent to P_2 at the point q_2, P_2 is tangent to P_3 at the point q_3, P_3 is tangent to P_4 at the point q_4, and P_4 is tangent to P_1 at the point q_1. Assume that all the points q_1, q_2, q_3, q_4 have distinct x-coordinates. Prove that q_1, q_2, q_3, q_4 lie on a graph of an at most quadratic polynomial.*

Solution We may subtract a quadratic trinomial from all the given trinomials so that the points q_1, q_2, q_3 get to the $0x$ axis. After this the trinomials remain trinomials, possibly degenerate, and the tangency is not affected. Let q_4' be the point of intersection of P_3 and $0x$ distinct from q_3; and let q_4'' be the intersection of P_4 and $0x$ distinct from q_1. If q_4' and q_4'' coincide then they also coincide with q_4 and the assertion follows.

Assume not. Every parabola (graph of a quadratic trinomial) that intersects the $0x$ axis twice, intersects it at the same angles. Having applied this to all the four parabolas in the circular order, we obtain that P_3 in q_4' and P_4 in $q_4'' \neq q_4'$ have the same slopes. But they also touch each other in the point q_4; therefore they are homothetic with respect to the point q_4. This homothety takes q_4' into q_4'', because the slope is preserved under the homothety and the point on a parabola is defined uniquely by its slope. Hence q_4 must also lie on the $0x$ axis, this is what we need to prove. The argument with homothety fails in one of the parabolas P_3 or P_4 degenerate to a straight line; but in this case the point q_4 also must be on $0x$ evidently. □