The 24th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 4th April 2014
Category I

Problem 1 *Find all complex numbers z such that $|z^3 + 2 - 2i| + z\bar{z}|z| = 2\sqrt{2}$. ($\bar{z}$ is the conjugate of z.)*

Problem 2 *We have a deck of $2n$ cards. Each shuffling changes the order from $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ to $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$. Determine all even numbers $2n$ such that after shuffling the deck 8 times the original order is restored.*

Problem 3 *Let $n \geq 2$ be an integer and let $x > 0$ be a real number. Prove that*

$$\left(1 - \sqrt{\tanh x}\right)^n + \sqrt{\tanh (nx)} < 1\,.$$

Recall that $\tanh t = \dfrac{e^{2t} - 1}{e^{2t} + 1}$.

Problem 4 *Let P_1, P_2, P_3, P_4 be the graphs of four quadratic polynomials drawn in the coordinate plane. Suppose that P_1 is tangent to P_2 at the point q_2, P_2 is tangent to P_3 at the point q_3, P_3 is tangent to P_4 at the point q_4, and P_4 is tangent to P_1 at the point q_1. Assume that all the points q_1, q_2, q_3, q_4 have distinct x-coordinates. Prove that q_1, q_2, q_3, q_4 lie on a graph of an at most quadratic polynomial.*