

The 23rd Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 12th April 2013
Category I

Problem 1 Let $f: [0, \infty) \rightarrow \mathbb{R}$ be a differentiable function with $|f(x)| \leq M$ and $f(x)f'(x) \geq \cos x$ for $x \in [0, \infty)$, where $M > 0$. Prove that $f(x)$ does not have a limit as $x \rightarrow \infty$.

Problem 2 Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real 10×10 matrices such that $a_{ij} = b_{ij} + 1$ for all i, j and $A^3 = 0$. Prove that $\det B = 0$.

Problem 3 Let S be a finite set of integers. Prove that there exists a number c depending on S such that for each non-constant polynomial f with integer coefficients the number of integers k satisfying $f(k) \in S$ does not exceed $\max(\deg f, c)$.

Problem 4 Let n and k be positive integers. Evaluate the following sum

$$\sum_{j=0}^k \binom{k}{j}^2 \binom{n+2k-j}{2k}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.