The 22^{nd} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 30^{th} March 2012
Category I

Problem 1 *Let $f\colon [0,1] \to [0,1]$ be a differentiable function such that $|f'(x)| \neq 1$ for all $x \in [0,1]$. Prove that there exist unique points $\alpha, \beta \in [0,1]$ such that $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.*

Solution *Existence:* Since f is derivable in $[0,1]$, then f is continuous in $[0,1]$. Considering the functions $g(x) = f(x) - x$ and $h(x) = f(x) - (1-x)$ that are continuous in $[0,1]$ and applying Bolzano's theorem we get that exists $\alpha \in [0,1]$ such that $g(\alpha) = 0$ and $\beta \in [0,1]$ with $h(\beta) = 0$. That is, there exist $\alpha, \beta \in [0,1]$ for which $f(\alpha) = \alpha$ and $f(\beta) = 1 - \beta$.

Uniqueness: Suppose that there exist $\alpha, \alpha' \in [0,1]$, $\alpha < \alpha'$) such that $f(\alpha) = \alpha$ and $f(\alpha') = \alpha'$. On account of Lagrange's theorem, there exists $\theta \in (\alpha, \alpha') \subset [0,1]$ such that

$$f'(\theta) = \frac{f(\alpha') - f(\alpha)}{\alpha' - \alpha} = \frac{\alpha' - \alpha}{\alpha' - \alpha} = 1$$

contradiction. Likewise, if we assume that there exist $\beta, \beta' \in [0,1]$, ($\beta < \beta'$) such that $f(\beta) = 1 - \beta$ and $f(\beta') = 1 - \beta'$. On account of Lagrange's theorem, there exists $\theta' \in (\beta, \beta') \subset [0,1]$ such that

$$f'(\theta') = \frac{f(\beta') - f(\beta)}{\beta' - \beta} = \frac{(1-\beta') - (1-\beta)}{\beta' - \beta} = -1$$

contradiction. This completes the proof. $\square$

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Problem 2 *Determine all 2×2 integer matrices A having the following properties:*

1. *the entries of A are (positive) prime numbers,*
2. *there exists a 2×2 integer matrix B such that $A = B^2$ and the determinant of B is the square of a prime number.*

Solution Let

$$A = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_4 \end{pmatrix} = B^2,$$

and $d = \det(B) = q^2$ with $p_1, p_2, p_3, p_4, q \in \mathbb{P}$; here $\mathbb{P}$ denotes the set of positive prime numbers.

By Cayley-Hamilton Theorem,

$$B^2 = \operatorname{tr}(B)B - \det(B)E,$$

where E is the 2×2 identity matrix. Without loss of generality, we assume that $\operatorname{tr}(B) \geq 0$, otherwise, replace B by $-B$. The equality

$$\operatorname{tr}(B)B = B^2 + dE = A + dE = \begin{pmatrix} p_1 + d & p_2 \\ p_3 & p_4 + d \end{pmatrix}$$

implies that $\operatorname{tr}(B)$ divides the numbers p_2 and p_3. Moreover,

$$(\operatorname{tr}(B))^2 = \operatorname{tr}(\operatorname{tr}(B)B) = p_1 + p_4 + 2d \geq 2 + 2 + 8 = 12 \implies \operatorname{tr}(B) > 3.$$

It follows that

$$\operatorname{tr}(B) = p_2 = p_3, \quad \text{and} \quad B = \frac{1}{\operatorname{tr}(B)} \begin{pmatrix} p_1 + d & p_2 \\ p_3 & p_4 + d \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$

for some positive integers a and b. Hence,

$$A = B^2 = \begin{pmatrix} a^2 + 1 & a + b \\ a + b & b^2 + 1 \end{pmatrix}.$$

The numbers $a^2 + 1, b^2 + 1, a + b$ cannot all be odd, thus, one of them equals 2. Since $ab = d + 1 = q^2 + 1 \geq 5$ we have $\max(a, b) \geq 3$. Hence, $a + b \geq 3 + 1 > 2$.

Now we assume that $a^2 + 1 \leq b^2 + 1$. Then $a^2 + 1 = 2$ and $a = 1$. Note that $d = ab - 1 = b - 1$ and $0 < p_2 = a+b = b+1 = d+2 = q^2+2$. If $q \neq 3$ then $q^2 \equiv 1 \mod 3 \implies p_2 \equiv 0 \mod 3 \implies p_2 = 3 \implies q^2 = 1$, which is impossible. Hence, $q = 3, b = p_2 - a = 3^2 + 2 - 1 = 10$,

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix}, \quad \text{and} \quad A = B^2 = \begin{pmatrix} 2 & 11 \\ 11 & 101 \end{pmatrix}.$$

Similarly, if $a^2 + 1 > b^2 + 1$ we obtain the matrix

$$A = \begin{pmatrix} 101 & 11 \\ 11 & 2 \end{pmatrix}.$$

Answer:

$$A = \begin{pmatrix} 2 & 11 \\ 11 & 101 \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 101 & 11 \\ 11 & 2 \end{pmatrix}.$$

$\square$

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Problem 3 *Determine the smallest real number C such that the inequality*

$$\frac{x}{\sqrt{yz}} \cdot \frac{1}{x+1} + \frac{y}{\sqrt{zx}} \cdot \frac{1}{y+1} + \frac{z}{\sqrt{xy}} \cdot \frac{1}{z+1} \le C$$

holds for all positive real numbers x, y and z with

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 1\,.$$

Solution In what follows we shall deal with the harder version of the problem only.

1. We consider the case $x = y = t$. Then
$$\frac{2}{t+1} + \frac{1}{z+1} = 1$$
that is
$$z = \frac{2}{t-1}.$$
Thus the inequality under consideration becomes
$$\frac{2t}{\sqrt{t \cdot \dfrac{2}{t-1}}} \cdot \frac{1}{t+1} + \frac{\frac{2}{t-1}}{\sqrt{t \cdot t}} \cdot \frac{1}{\frac{2}{t-1}+1} \le C$$
that is
$$\frac{\sqrt{2} \cdot \sqrt{t} \cdot \sqrt{t-1}}{t+1} + \frac{2}{t(t+1)} \le C.$$
Letting here $t \to \infty$ leads to $C \ge \sqrt{2}$.

2. We are now going to prove that always
$$\frac{x}{\sqrt{yz}} \cdot \frac{1}{x+1} + \frac{y}{\sqrt{zx}} \cdot \frac{1}{y+1} + \frac{z}{\sqrt{xy}} \cdot \frac{1}{z+1} < \sqrt{2}.$$
In order to achieve this goal we make use of the following transformation
$$a = \frac{1}{x+1}, \quad b = \frac{1}{y+1}, \quad c = \frac{1}{z+1}.$$
Then the three new variables satisfy a, b, $c \in (0;1)$ and are subject to the condition $a+b+c=1$. Furthermore
$$x = \frac{1-a}{a}, \quad y = \frac{1-b}{b}, \quad z = \frac{1-c}{c}$$
that is (due to $1-a = b+c$, etc.)
$$x = \frac{b+c}{a}, \quad y = \frac{c+a}{b}, \quad z = \frac{a+b}{c}$$
yield for the claimed inequality
$$\frac{(a+b)\sqrt{ab}}{\sqrt{(b+c)(c+a)}} + \frac{(b+c)\sqrt{bc}}{\sqrt{(c+a)(a+b)}} + \frac{(c+a)\sqrt{ca}}{\sqrt{(a+b)(b+c)}} < \sqrt{2}.$$
Upon clearing fractions this inequality becomes
$$(a+b)\sqrt{ab(a+b)} + (b+c)\sqrt{bc(b+c)} + (c+a)\sqrt{ca(a+c)} < \sqrt{2(a+b)(b+c)(c+a)}.$$

We smuggle the condition $1 = a + b + c$ into the inequality and get

$$(a+b)\sqrt{ab(a+b)} + (b+c)\sqrt{bc(b+c)} + (c+a)\sqrt{ca(a+c)} < \sqrt{2(a+b)(b+c)(c+a)}(a+b+c).$$

Next, we deal with the right-hand expressions. For them we have

$$\sqrt{(a+b)(b+c)(c+a)} = \sqrt{ab(a+b) + bc(b+c) + ca(c+a) + 2abc}$$

and

$$\sqrt{2}(a+b+c) = \sqrt{2(a+b+c)^2} = \sqrt{(a+b)^2 + (a+c)^2 + (b+c)^2 + 2(ab+bc+ca)}$$

But, employing the Cauchy-Schwarz inequality yields for our inequality

$$(a+b)\sqrt{ab(a+b)} + (b+c)\sqrt{bc(b+c)} + (c+a)\sqrt{ca(a+c)} \leq \sqrt{(a+b)^2 + (b+c)^2 + (c+a)^2} \cdot \sqrt{ab(a+b) + ac(a+c) + bc(b+c)}.$$

This together with the two previously stated equations completes the proof. It is also evident that there cannot exist any triples (a, b, c), and thus also (x, y, z), yielding equality.

$\square$

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Problem 4 *Find all positive integers n for which there exists a positive integer k such that the decimal representation of n^k starts and ends with the same digit.*

Solution The number n^k ends with zero whenever n is divisible by 10 and starts with nonzero digit. We show that the claim is true for all other n's.

It can be easily shown that all the numbers

$$n, n^5, n^9, \ldots, n^{4m+1}, \ldots \tag{1}$$

ends with the same digit. In fact, $n^5 - n = n(n-1)(n+1)(n^2+1)$ is even and for each possible reminder of n modulo 5 there is a factor divisible by 5 in this product. Thus $n^5 - n$ is divisible by 10 and in the same fashion we can show this for $n^9 - n^5, n^{13} - n^9, \ldots$

Now it suffices to show that for any nonzero digit c there is a number in the sequence (1) which starts with c. For any nonnegative integer m put $d_m = n^{4m+1}/10^l$, where l is the greatest integer for which $10^l \le n^{4m+1}$. Thus $1 \le d_m < 10$ and $\lfloor d_m \rfloor$ is the first digit of n^{4m+1}. Clearly all the d_m's are different, since for $m' > m$ we have

$$\frac{d_{m'}}{d_m} = \frac{n^{4m'+1}/10^{l'}}{n^{4m+1}/10^l} = \frac{n^{4(m'-m)}}{10^{l'-l}} \ne 1$$

(the numerator is not a power of 10 for n not divisible by 10).

The sequence $(d_m)_{m=1}^{\infty}$ has the following property: If $d_{m+i} = d_m \cdot q$, then $d_{m+2i} = d_m \cdot q^2 \cdot 10^{\varepsilon}$, where $\varepsilon \in \{-1, 0, 1\}$. This is true since when

$$d_m = n^{4m+1}/10^l, \quad d_{m+i} = n^{4(m+i)+1}/10^{l'} \quad \text{and} \quad d_{m+2i} = n^{4(m+2i)+1}/10^{l''},$$

we have $q = d_{m+i}/d_m = n^{4i}/10^{l'-l}$ and so

$$d_{m+2i}/d_m = n^{8i}/10^{l''-l} = q^2 \cdot 10^{2l'-l-l''} = q^2 \cdot 10^{\varepsilon}$$

for some integer ε. But $d_m, d_m \cdot q, d_m \cdot q^2 \cdot 10^{\varepsilon} \in [1, 10)$, i.e. $\varepsilon \in \{-1, 0, 1\}$.

Since all the terms of the sequence $(d_m)_{m=1}^{\infty}$ are different and all lie in the interval $[1, 10)$, there have to be two terms d_m and $d_{m'}$ such that $|d_{m'} - d_m| < \frac{1}{10}$. Without loss of generality let $m' > m$. There are two possibilities.

Let $d_{m'} > d_m$. Then we have $d_{m'} < d_m + \frac{1}{10}$. Thus

$$1 < q = d_{m'}/d_m < \frac{d_m + \frac{1}{10}}{d_m} = 1 + \frac{1}{10 d_m} \le 1 + \tfrac{1}{10}.$$

By previous remark $d_m \cdot q^2$ lies in the studied sequence, whenever it lies in the interval $[1, 10)$. Repeating this idea we have the numbers $d_m, d_m \cdot q, d_m \cdot q^2, d_m \cdot q^3, \ldots, d_m \cdot q^i$ all lying in the studied sequence and after overrunning the value 10 we have the numbers $d_m \cdot q^{i+1}/10, d_m \cdot q^{i+2}/10, \ldots$ in the sequence, and so on. Computing the difference of two consecutive terms in this recurrence we get

$$d_m \cdot q^{j+1} - d_m \cdot q^j = d_m \cdot q^j (q-1) < d_m \cdot q^j \cdot \tfrac{1}{10} < 1 \qquad \text{for } j < i,$$
$$d_m \cdot q^{j+1}/10 - d_m \cdot q^j/10 = d_m \cdot q^j (q-1)/10 < d_m \cdot q^j/10 \cdot \tfrac{1}{10} < 1 \qquad \text{for } j > i$$

and for the first term after overrunning 10 we obtain

$$d_m \cdot q^{i+1}/10 = d_m \cdot q^i/10 \cdot q < 10/10 \cdot (1 + \tfrac{1}{10}) = \tfrac{11}{10} < 2.$$

Since the difference is less then 1 and after overrunning we jump into the interval $[1, 2)$, we must get at least one $d_{m+j(m'-m)}$ in the interval $[c, c+1)$ for every nonzero digit c.

Let $d_{m'} < d_m$. Then we have $d_m < d_{m'} + \frac{1}{10}$. Thus

$$1 < q = d_m/d_{m'} < \frac{d_{m'} + \frac{1}{10}}{d_{m'}} = 1 + \frac{1}{10 d_{m'}} \le 1 + \tfrac{1}{10}.$$

In the very similar way as in the previous case (new terms are generated by dividing instead of multiplying by q) we obtain the new sequence of terms with consecutive differences less then 1 and after underrunning 1 we jump to

$$d_m/q^{i+1} \cdot 10 = d_m/q^i \cdot 10/q > 1 \cdot \frac{10}{1+\frac{1}{10}} = \tfrac{100}{11} > 9.$$

Thus also in this case we must obtain some $d_{m+j(m'-m)}$ in the interval $[c, c+1)$ for every nonzero digit c.

This ends the proof.

Answer. Integers satisfying the given conditions are all integers not divisible by 10. $\square$