The 20th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 25th March 2010
Category I

Problem 1

a) *Is it true that for every bijection* $f\colon \mathbb{N}\to\mathbb{N}$ *the series*

$$\sum_{n=1}^{\infty}\frac{1}{nf(n)}$$

is convergent?

b) *Prove that there exists a bijection* $f\colon \mathbb{N}\to\mathbb{N}$ *such that the series*

$$\sum_{n=1}^{\infty}\frac{1}{n+f(n)}$$

is convergent.

($\mathbb{N}$ is the set of all positive integers.) [10 points]

Solution a) Yes. Applying the inequality, if $0\le a_1\le\cdots\le a_n$ and $0\le b_1\le\cdots\le b_n$ and $\sigma\colon\{1,\dots,n\}\to\{1,\dots,n\}$ is a permutation, then

$$\sum_{j=1}^{n}a_jb_{\sigma(j)}\le\sum_{j=1}^{n}a_jb_j,$$

for every n we get

$$\sum_{j=1}^{n}\frac{1}{jf(j)}\le\sum_{j=1}^{n}\frac{1}{j^2}\le\sum_{j=1}^{\infty}\frac{1}{j^2}.$$

Since the sequence $\left(\sum_{j=1}^{n}\frac{1}{jf(j)}\right)$ is increasing and bounded, it converges.

b) No. We will construct a permutation $f\colon\mathbb{N}\to\mathbb{N}$ such that the series

$$\sum_{n=1}^{\infty}\frac{1}{n+f(n)}$$

is convergent. Let $f\colon\mathbb{N}\to\mathbb{N}$ be given in the following way: $f(1)=4$ and for $[(n!)^2+1,((n+1)!)^2]\cap\mathbb{N}$ we put

$$f((n!)^2+k)=[(n+2)!]^2-(k-1)\quad\text{if}\quad 1\le k<[(n+1)!]^2-1-\sum_{j=0}^{n-1}(-1)^j[(n-j)!]^2.$$

and

$$f([(n+1)!]^2-k)=[(n-1)!]^2+k+1\quad\text{if}\quad 0\le k\le 1+\sum_{j=0}^{n-1}(-1)^j[(n-j)!]^2.$$

Then

$$\sum_{j=(n!)^2+1}^{[(n+1)!]^2}\frac{1}{n+f(n)}\le\frac{((n+1)!)^2-(n!)^2}{(n!)^2+[(n+2)!]^2+1}+\frac{(n!)^2-[(n-1)!]^2}{[(n+1)!]^2+[(n-1)!]^2+1}$$
$$<\frac{1}{(n+2)^2}+\frac{1}{(n+1)^2}.$$

Thus we show that the sequence $\left(\sum_{j=1}^{n}\frac{1}{j+f(j)}\right)$ is bounded. Since it is increasing, it converges. □

Problem 2 *Let A and B be two complex 2×2 matrices such that $AB - BA = B^2$. Prove that $AB = BA$.*

[10 points]

Solution We may conclude that $AB = BA$ if and only if $2 \neq 0$ in F (that is, $\operatorname{char} F \neq 2$).

If $\operatorname{char} F = 2$, take $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Assume that $\operatorname{char} F \neq 2$. Let $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $B^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$. We have $a^2 + d^2 + 2bc = \operatorname{trace} B^2 = \operatorname{trace} AB - \operatorname{trace} BA = 0$. If B is invertible, then $A = B(A+B)B^{-1}$, hence

$$\operatorname{trace} A = \operatorname{trace}(B(A+B)B^{-1}) = \operatorname{trace}(A+B) = \operatorname{trace} A + \operatorname{trace} B,$$

so $\operatorname{trace} B = 0$, $d = -a$, $\operatorname{trace} B^2 = 2(a^2 + bc) = 0$. Since $\operatorname{char} F \neq 2$, it implies $a^2 + bc = 0$, hence $B^2 = 0$ and $AB = BA$. If B is not invertible, then $\det B = ad - bc = 0$, so $(a+d)^2 = a^2 + d^2 + 2bc = 0$, $a + d = 0$, $a = -d$, $a^2 + bc = -ad + bc = 0$, so $B^2 = 0$. □

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Problem 3 *Prove that there exist positive constants c_1 and c_2 with the following properties:*

a) For all real $k > 1$,

$$\left|\int_0^1 \sqrt{1-x^2}\,\cos(kx)\,\mathrm{d}x\right| < \frac{c_1}{k^{3/2}}\,.$$

b) For all real $k > 1$,

$$\left|\int_0^1 \sqrt{1-x^2}\,\sin(kx)\,\mathrm{d}x\right| > \frac{c_2}{k}\,.$$

[10 points]

Solution Put $f(x) = \sqrt{1-x^2}$.

1. Integrating by parts, we have

$$\int_0^1 f(x)\cdot\cos kx\,\mathrm{d}x = \Big[f(x)\cdot\frac{1}{k}\sin kx\Big]_0^1 - \int_0^1 f'(x)\cdot\frac{1}{k}\sin kx\,\mathrm{d}x\,.$$

The first term is $0-0=0$. The second term is $(-1/k)$ times

$$\int_0^{\sqrt{1-1/k}} f'(x)\cdot\sin kx\,\mathrm{d}x + \int_{\sqrt{1-1/k}}^1 f'(x)\cdot\sin kx\,\mathrm{d}x\,. \tag{1}$$

Here the first term equals

$$\Big[-f'(x)\cdot\frac{1}{k}\cos kx\Big]_0^{\sqrt{1-1/k}} + \int_0^{\sqrt{1-1/k}} f''(x)\cdot\frac{1}{k}\cos kx\,\mathrm{d}x\,,$$

whose absolute value is

$$\le -\frac{2}{k}f'\big(\sqrt{1-1/k}\big) = \frac{2}{k}\frac{\sqrt{1-1/k}}{\sqrt{1/k}} < \frac{2}{\sqrt{k}}\,.$$

The absolute value of the second term in (1) is

$$\le \int_{\sqrt{1-1/k}}^1 |f'(x)|\,\mathrm{d}x = -[f(x)]_{\sqrt{1-1/k}}^1 = \frac{1}{\sqrt{k}}\,.$$

Thus, we may choose $c_1 = 2+1 = 3$.

2. Integrating by parts, we have

$$\int_0^1 f(x)\cdot\sin kx\,\mathrm{d}x = -\Big[f(x)\cdot\frac{1}{k}\cos kx\Big]_0^1 + \int_0^1 f'(x)\cdot\frac{1}{k}\cos kx\,\mathrm{d}x\,.$$

The first term is $1/k$. The second term is $(1/k)$ times

$$\int_0^{\sqrt{1-1/k}} f'(x)\cdot\cos kx\,\mathrm{d}x + \int_{\sqrt{1-1/k}}^1 f'(x)\cdot\cos kx\,\mathrm{d}x\,. \tag{2}$$

Here the first term equals

$$\Big[f'(x)\cdot\frac{1}{k}\sin kx\Big]_0^{\sqrt{1-1/k}} - \int_0^{\sqrt{1-1/k}} f''(x)\cdot\frac{1}{k}\sin kx\,\mathrm{d}x\,,$$

whose absolute value is

$$\le -\frac{2}{k}f'\big(\sqrt{1-1/k}\big) = \frac{2}{k}\frac{\sqrt{1-1/k}}{\sqrt{1/k}} < \frac{2}{\sqrt{k}}\,.$$

The absolute value of the second term in (2) is

$$\le \int_{\sqrt{1-1/k}}^1 |f'(x)|\,\mathrm{d}x = -[f(x)]_{\sqrt{1-1/k}}^1 = \frac{1}{\sqrt{k}}\,.$$

Thus,

$$\int_0^1 f(x) \cdot \sin kx \,\mathrm{d}x > \frac{1}{k}\left(1 - \frac{3}{\sqrt{k}}\right).$$

This proves the desired claim for $k \geq 3\pi$.

The integral has a positive lower bound for $k < 3\pi$ as well, since

$$\int_0^1 f(x) \cdot \sin kx \,\mathrm{d}x = \int_0^1 \big(-f'(x)\big) \cdot \frac{1 - \cos kx}{k} \,\mathrm{d}x > 0\,.$$

□

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Problem 4 *For every positive integer n let $\sigma(n)$ denote the sum of all its positive divisors. A number n is called* weird *if $\sigma(n) \geq 2n$ and there exists no representation*

$$n = d_1 + d_2 + \cdots + d_r \,,$$

where $r > 1$ and $d_1, \ldots, d_r$ are pairwise distinct positive divisors of n.
Prove that there are infinitely many weird numbers. [10 points]

Solution The idea is to show that given a weird number, one can construct a sequence of weird numbers tending to infinity.

We claim that for weird n and p a prime greater than $\sigma(n)$ and coprime to n, the number pn is also weird. In fact, if $1 = d_1, d_2, \ldots, d_k = n$ are the positive divisors of n, the ones of pn are $d_1, d_2, \ldots, d_k, pd_1, \ldots, pd_k$ and they are pairwise distinct as $(p, n) = 1$. Suppose now that we have

$$pn = d_{i_1} + \cdots + d_{i_r} + p(d_{j_1} + \cdots + d_{j_s})$$

with $i_k, j_l \in \{1, \ldots, k\}$. Then we have

$$d_{i_1} + \cdots + d_{i_r} = p(n - d_{j_1} - \cdots - d_{j_s}) \,.$$

Note that $n \notin \{d_{j_1}, \ldots, d_{j_s}\}$ as the representation must have more than only one summand and the assumption that n is weird implies $n - d_{j_1} - \ldots - d_{j_s} \neq 0$. Hence as the right hand expression is divisible by p and non zero, so must be $d_{i_1} + \cdots + d_{i_r}$ which is impossible as $p > \sigma(n)$.

It remains to find a weird number. A possible reasoning could be: look for a number n with $\sigma(n) = 2n + 4$ that is not divisible by 3 and 4. Then the smallest possible divisors are $1, 2, 5$ so that it will be impossible to represent 4, and hence n, as a sum of pairwise distinct divisors of n. Checking for numbers with three distinct prime factors $2, p, q$ yields

$$\sigma(2pq) = 3(p + 1)(q + 1) = 3pq + 3p + 3q + 3$$

and hence we need

$$3pq + 3p + 3q + 3 = 4pq + 4 \iff (p - 3)(q - 3) = 8 \,.$$

This equality is solved by $p = 5$ and $q = 7$ which yields the weird number $n = 70$. □