Problem j18-I-1. *Find all complex roots (with multiplicities) of the polynomial*

$$p(x) = \sum_{n=1}^{2008} \bigl(1004 - |1004 - n|\bigr) x^n \, .$$

Solution. Observe, by comparison of coefficients, that

$$p(x) = x \Bigl(\sum_{n=0}^{1003} x^n \Bigr)^2 \, .$$

Since $\sum_{n=0}^{1003} x^n = \frac{x^{1004}-1}{x-1}$, we conclude that p has the simple root 0 and the roots $\exp \frac{\pi i n}{502}$, $n = 1, 2, \ldots, 1003$, with multiplicity 2. □

Problem j18-I-2. *Find all functions* $f\colon (0,\infty) \to (0,\infty)$ *such that*

$$f(f(f(x))) + 4f(f(x)) + f(x) = 6x\,.$$

Solution. Let $a \in \mathbb{R}^+$ be arbitrary. Set $a_0 = a$, $a_n = f(a_{n-1})$ for $n > 0$. Then we obtain recurrence relation

$$a_{n+3} + 4a_{n+2} + a_{n+1} - 6a_n = 0\,.$$

Characteristic equation is

$$y^3 - 4y^2 + y - 6 = 0$$

with roots -2, -3 and 1. The general solution of recurrence relation is

$$a_n = A(-3)^n + B(-2)^n + C\,.$$

If A or B are not equal to 0, we have a contradiction because in range of f we could find negative values. So the only possible solution is $a_n = C$. Because of $a_0 = a$ we have $a_n = a$ for all $n \in \mathbb{N}_0$. Substituting $n = 1$ we obtain

$$f(a) = f(a_0) = a_1 = a\,,$$

so for all $a \in \mathbb{R}^+$ we have $f(a) = a$.

The only solution of the equation is $f(x) = x$, what can be easily checked. $\square$

Problem j18-I-3. *Find all $c \in \mathbb{R}$ for which there exists an infinitely differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have*

$$f^{(n+1)}(x) > f^{(n)}(x) + c\,. \tag{1}$$

Solution. For $c \le 0$ we can take $f(x) = \mathrm{e}^{2x}$. Then $f^{(n+1)}(x) = 2^{n+1}\mathrm{e}^{2x} > 2^n\mathrm{e}^{2x} = f^{(n)}(x)$.

For positive c no function satisfies (1). We begin with two simple lemmas.

Lemma 1. *If f satisfies* (1)*, then for any $x \in \mathbb{R}$ there exists an $y \le x$ such that $f(y) \le -\frac{c}{2}$.*

Proof. If $f(t) > -\frac{c}{2}$ on $(-\infty, x]$, then $f'(t) > \frac{c}{2}$ for any $t < x$, thus

$$f(y) = f(x) - \int_y^x f'(t)\,\mathrm{d}t \le f(x) - (x-y)\frac{c}{2}$$

for any $y < x$, thus for sufficiently small y we have $f(y) < 0$, a contradiction. ▫

Lemma 2. *If f satisfies* (1)*, then for any $x \in \mathbb{R}$ such that $f(x) < \frac{c}{2}$ we have $f(y) < \frac{c}{2}$ for any $y \le x$.*

Proof. Suppose that there exists a $y \le x$ such that $f(y) \ge -\frac{c}{2}$. Let $z := \sup\{t \le x : f(t) \ge -\frac{c}{2}\}$. By the continuity of f (f is differentiable, thus continuous) we have $f(z) \ge -\frac{c}{2}$. By the assumption upon x we have $z \ne x$. However by (1) we have $f'(z) \ge \frac{c}{2}$, thus f' is positive on $[z, z+\varepsilon]$ for some $\varepsilon > 0$, f is increasing, thus $f(t) \ge f(z) \ge -\frac{c}{2}$ for $t \in [z, z+\varepsilon]$, a contradiction with the definition of z. Thus by contradiction the thesis is proved. ▫

Now if f satisfies (1), then obviously f' also satisfies (1). Thus by Lemmas 1 and 2, there exists an x_0 such that $f'(t) < -\frac{c}{2}$ on $(-\infty, x_0]$. This, however, means $f(t) > f(x_0) + (x_0 - t)\frac{c}{2}$ for $t < x_0$, so for sufficiently small $t_0 < x_0$ we have $f(t_0) > -\frac{3c}{2} > f'(t_0) - c$, which is a contradiction with (1). Thus no such f exists. □

Problem j18-I-4. *The numbers of the set $\{1,2,\ldots,n\}$ are colored with 6 colors. Let*

$$S := \big\{(x,y,z) \in \{1,2,\ldots,n\}^3 : x+y+z \equiv 0 \pmod n \\ \text{and } x,y,z \text{ have the same color}\big\}$$

and

$$D := \big\{(x,y,z) \in \{1,2,\ldots,n\}^3 : x+y+z \equiv 0 \pmod n \\ \text{and } x,y,z \text{ have three different colors}\big\}.$$

Prove that

$$|D| \le 2|S| + \frac{n^2}{2}.$$

(For a set A, $|A|$ denotes the number of elements in A.)

Solution. Denote by n_1,n_2,n_3,n_4,n_5,n_6 the number of occurences of the colors. Clearly $n_1+\ldots+n_6 = n$. We prove that

$$|S| - \frac{1}{2}|D| = \sum_{u=1}^{6} n_u^2 - \sum_{1\le u<v\le 6} n_u n_v. \tag{1}$$

For arbitrary $u,v,w \in \{1,2,\ldots,6\}$, denote by N_{uvw} the number of triples (x,y,z), satisfying $x+y+z\equiv 0 \pmod n$ and having colors u, v and w, respectively. For any u,v we obviously have $\sum\limits_{w=1}^{6} N_{uvw} = n_u n_v$ and therefore

$$\begin{aligned}
|S| - \frac{1}{2}|D| &= \sum_{u=1}^{6} N_{uuu} - \sum_{1\le u<v\le 6}\sum_{w\ne u,v} N_{uvw} \\
&= \sum_{u=1}^{6}\Big(n_u^2 - \sum_{v\ne u} N_{uuv}\Big) - \sum_{1\le u<v\le 6}(n_u n_v - N_{uuv} - N_{uvv}) \\
&= \sum_{u=1}^{6} n_u^2 - \sum_{1\le u<v\le 6} n_u n_v.
\end{aligned}$$

Now, applying the AM-QM inequality,

$$\begin{aligned}
|S| - \frac{1}{2}|D| &= \sum_{u=1}^{6} n_u^2 - \sum_{1\le u<v\le 6} n_u n_v = \frac{3}{2}\sum_{u=1}^{6} n_u^2 - \frac{1}{2}\Big(\sum_{u=1}^{6} n_u\Big)^2 \\
&\ge \Big(\frac{1}{4} - \frac{1}{2}\Big)\Big(\sum_{u=1}^{6} n_u\Big)^2 = -\frac{n^2}{4}.
\end{aligned}$$

□

Second solution. We present a different proof for the relation (1). We use the notation N_{uvw} as well.

For every $u = 1,2,\ldots,6$, let C_u be the set of those numbers from $\{1,2,\ldots,n\}$ which have the uth color and let $f_u(t) := \sum\limits_{x\in C_u} t^x$.

Let $\varepsilon := e^{2\pi i/n}$. We will use that for every integer s,

$$\frac{1}{n}\sum_{j=0}^{n-1}\varepsilon^{js} = \begin{cases} 1 & \text{if } s\equiv 0 \pmod n \\ 0 & \text{if } s\not\equiv 0 \pmod n\,. \end{cases}$$

Then, for arbitrary colors u,v,w,

$$\begin{aligned}
N_{uvw} &= \sum_{x\in C_u}\sum_{y\in C_v}\sum_{z\in C_w}\frac{1}{n}\sum_{j=0}^{n-1}\varepsilon^{j(x+y+z)} \\
&= \frac{1}{n}\sum_{j=0}^{n-1}\Big(\sum_{x\in C_u}\varepsilon^{jx}\Big)\Big(\sum_{y\in C_v}\varepsilon^{jy}\Big)\Big(\sum_{z\in C_w}\varepsilon^{jz}\Big) = \frac{1}{n}\sum_{j=0}^{n-1} f_u(\varepsilon^j) f_v(\varepsilon^j) f_w(\varepsilon^j)
\end{aligned}$$

and

$$|S| - \frac{1}{2}|D| = \frac{1}{n}\sum_{j=0}^{n-1}\Big(\sum_{u=1}^{6} f_u^3(\varepsilon^j) - 3\sum_{u<v<w} f_u(\varepsilon^j)f_v(\varepsilon^j)f_w(\varepsilon^j)\Big)$$

$$= \frac{1}{n}\sum_{j=0}^{n-1}\Big(\sum_{u=1}^{6} f_u(\varepsilon^j)\Big)\Big(\sum_{u=1}^{6} f_u^2(\varepsilon^j) - \sum_{u<v} f_u(\varepsilon^j)f_v(\varepsilon^j)\Big)$$

$$= \sum_{j=0}^{n-1}\Big(\frac{1}{n}\sum_{x=1}^{n}\varepsilon^{jx}\Big)\Big(\sum_{u=1}^{6} f_u^2(\varepsilon^j) - \sum_{u<v} f_u(\varepsilon^j)f_v(\varepsilon^j)\Big)\,.$$

The first factor is 0 except if $j = 0$. Hence,

$$|S| - \frac{1}{2}|D| = \sum_{u=1}^{6} f_u^2(1) - \sum_{u<v} f_u(1)f_v(1) = \sum_{u=1}^{6} n_u^2 - \sum_{u<v} n_u n_v\,.$$

□