The 18th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2008
Category I

Problem 1. *Find all complex roots (with multiplicities) of the polynomial*

$$p(x) = \sum_{n=1}^{2008} \bigl(1004 - |1004 - n|\bigr) x^n \,.$$

[10 points]

Problem 2. *Find all functions* $f\colon (0,\infty) \to (0,\infty)$ *such that*

$$f(f(f(x))) + 4f(f(x)) + f(x) = 6x \,.$$

[10 points]

Problem 3. *Find all* $c \in \mathbb{R}$ *for which there exists an infinitely differentiable function* $f\colon \mathbb{R} \to \mathbb{R}$ *such that for all* $n \in \mathbb{N}$ *and* $x \in \mathbb{R}$ *we have*

$$f^{(n+1)}(x) > f^{(n)}(x) + c \,.$$

[10 points]

Problem 4. *The numbers of the set* $\{1, 2, \ldots, n\}$ *are colored with* 6 *colors. Let*

$$S := \bigl\{(x,y,z) \in \{1,2,\ldots,n\}^3 : x+y+z \equiv 0 \pmod{n} \\ \text{and } x,y,z \text{ have the same color}\bigr\}$$

and

$$D := \bigl\{(x,y,z) \in \{1,2,\ldots,n\}^3 : x+y+z \equiv 0 \pmod{n} \\ \text{and } x,y,z \text{ have three different colors}\bigr\} \,.$$

Prove that

$$|D| \le 2|S| + \frac{n^2}{2} \,.$$

(For a set A*,* $|A|$ *denotes the number of elements in* A*.)* [10 points]