The 16th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 29th March 2006
Category I

Problem 1. Given real numbers $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$ such that $x_{i+1} - x_i \leq h$ for $1 \leq i \leq 2n$, show that

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) < \frac{1+h}{2}.$$

[10 points]

Problem 2. Suppose that (a_n) is a sequence of real numbers such that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

is convergent. Show that the sequence

$$b_n = \frac{\sum_{j=1}^{n} a_j}{n}$$

is convergent and find its limit. [10 points]

Problem 3. Two players play the following game: Let n be a fixed integer greater than 1. Starting from number $k = 2$, each player has two possible moves: either replace the number k by $k+1$ or by $2k$. The player who is forced to write a number greater than n loses the game. Which player has a winning strategy for which n? [10 points]

Problem 4. Let $A = [a_{ij}]_{n \times n}$ be a matrix with nonnegative entries such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = n.$$

(a) Prove that $|\det A| \leq 1$.

(b) If $|\det A| = 1$ and $\lambda \in \mathbb{C}$ is an arbitrary eigenvalue of A, show that $|\lambda| = 1$.

(We call $\lambda \in \mathbb{C}$ an eigenvalue of A if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$.) [10 points]