Problem j12-I-1/j12-I-4. *Differentiable functions $f_1, \dots, f_n \colon \mathbb{R} \to \mathbb{R}$ are linearly independent. Prove that there exist at least $n-1$ linearly independent functions among $f_1', \dots, f_n'$.*

(Eötvös Loránd University, Budapest)

Solution. Select a maximal independent set from the derivatives. Without loss of generality, it can be assumed that this set is $f_1', \dots, f_m'$, where $m \le n$. If $m \le n-2$, then f_{n-1}' and f_n' can be expressed as a linear combination of $f_1', \dots, f_m'$; hence, there exist real numbers $a_1, \dots, a_m$, $b_1, \dots, b_m$ such that

$$\sum_{i=1}^{m} a_i f_i' - f_{n-1}' = \left(\sum_{i=1}^{m} a_i f_i - f_{n-1} \right)' = 0$$

and

$$\sum_{i=1}^{m} b_i f_i' - f_n' = \left(\sum_{i=1}^{m} b_i f_i - f_n \right)' = 0.$$

This implies that functions $\sum_{i=1}^{m} a_i f_i - f_{n-1}$ and $\sum_{i=1}^{m} b_i f_i - f_n$ are constant. Eliminating these constants, a linear combination of $f_1, \dots, f_n$ is found which vanishes. $\square$

Problem j12-I-2/j12-I-9. *Let $p > 3$ be a prime number and $n = \frac{2^{2p}-1}{3}$. Show that n divides $2^n - 2$.* (Jagiellonian University in Kraków)

Solution. $n = \frac{2^{2p}-1}{3} = 4^{p-1}+4^{p-2}+\cdots+1$. Hence, in the binary system, $n = 1010\ldots101$ (number of 1's is p). Therefore, in the binary system,

$$(*)\qquad 2^n - 2 = 1111\ldots110 \qquad \text{(number of 1's is } n-1\text{)},$$

$$(**)\qquad 3n = 1111\ldots111 \qquad \text{(number of 1's is } 2p\text{)}.$$

Now if we prove that $2p$ divides $n-1$, then by $(*)$, $(**)$ and by the rules of multiplication in the binary system, we will get that $3n$ divides $2^n - 2$ — just what we need. But now observe:

$$\begin{aligned} 2p \mid (n-1) &\iff (n \text{ is odd}\dagger) \iff p \mid (n-1) \iff \\ &\iff p \,\Big|\, \left(\frac{2^{2p}-1}{3} - 1\right) \iff p \,\Big|\, \left(\frac{2^{2p}-4}{3}\right) \iff \\ &\iff (p > 3 \text{ and prime}) \iff p \mid \left(2^{2p}-4\right) \iff \\ &\iff (p > 3 \text{ and prime}) \iff p \,\Big|\, \left(\frac{2^{2p}-4}{4}\right) \iff \\ &\iff p \mid \left(2^{2p-2}-1\right). \end{aligned}$$

But now from Fermat's small theorem (p prime and p does not divide a, then $a^{p-1} - 1 \equiv 0 \pmod{p}$), we have $2^{p-1} \equiv 1 \pmod{p}$, hence $(2^{p-1})^2 \equiv 1^2 \pmod{p}$ and finally $2^{2p-2} \equiv 1 \pmod{p}$. □

† The sentences in parentheses serve only as justifications of the stated equivalences here. Thus, e.g., $2p \mid (n-1) \Leftrightarrow (n \text{ is odd}) \Leftrightarrow p \mid (n-1)$ should be read as "$2p$ divides $(n-1)$ if and only if p divides $(n-1)$ because n is odd" and so on.

Problem j12-I-3/j12-II-59. *Positive numbers* $x_1, \ldots, x_n$ *satisfy*

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + \cdots + \frac{1}{1+x_n} = 1. \tag{1}$$

Prove that

$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \geq (n-1)\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right).$$

(University of Ostrava)

Solution. It is sufficient to prove that

$$\left(\sqrt{x_1} + \frac{1}{\sqrt{x_1}}\right) + \left(\sqrt{x_2} + \frac{1}{\sqrt{x_2}}\right) + \cdots + \left(\sqrt{x_n} + \frac{1}{\sqrt{x_n}}\right) \geq n\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right)$$

or equivalently (see (1))

$$\left(\frac{1+x_1}{\sqrt{x_1}} + \cdots + \frac{1+x_n}{\sqrt{x_n}}\right)\left(\frac{1}{1+x_1} + \cdots + \frac{1}{1+x_n}\right) \geq n\left(\frac{1}{\sqrt{x_1}} + \frac{1}{\sqrt{x_2}} + \cdots + \frac{1}{\sqrt{x_n}}\right). \tag{2}$$

Consider the function $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} = \frac{x+1}{\sqrt{x}}$, $x \in (0, +\infty)$. It is easy to check that the function f is non-decreasing on $[1, +\infty)$ and that

$$f(x) = f(\tfrac{1}{x}) \tag{3}$$

holds for each $x > 0$.

Further, it follows from (1) that only x_1 can be less than 1 (i.e. $x_k \geq 1$, $k = 2, 3, \ldots$) and $\frac{1}{1+x_2} \leq 1 - \frac{1}{1+x_1} = \frac{x_1}{1+x_1}$. Hence

$$x_2 \geq \frac{1}{x_1} \tag{4}$$

(a contradiction otherwise). It is now apparent directly (if $x_1 \geq 1$) or from (3) and (4) (if $x_1 < 1$) that

$$f(x_1) = f(\tfrac{1}{x_1}) \leq f(x_2) \leq \cdots \leq f(x_n).$$

This means that the sequence $\left\{\frac{1+x_k}{\sqrt{x_k}}\right\}_{k=1}^{n}$ is non-decreasing. Thus (2) holds according to the well-known Chebyshev's inequality since the sequence $\left\{\frac{1}{1+x_k}\right\}_{k=1}^{n}$ is decreasing.

The equality in (2) holds if and only if

$$\frac{1}{1+x_1} = \frac{1}{1+x_2} = \cdots = \frac{1}{1+x_n} \qquad \text{or} \qquad \frac{1+x_1}{\sqrt{x_1}} = \frac{1+x_2}{\sqrt{x_2}} = \cdots = \frac{1+x_n}{\sqrt{x_n}},$$

which implies $x_1 = x_2 = \cdots = x_n$. Then we obtain from (1) that $x_1 = x_2 = \cdots = x_n = n-1$. $\square$

Problem j12-I-4/j12-I-5. *The numbers* $1, 2, \ldots, n$ *are assigned to the vertices of a regular* n*-gon in an arbitrary order. For each edge compute the product of the two numbers at the endpoints and sum up these products. What is the smallest possible value of this sum?*

(Babeş-Bolyai University, Cluj-Napoca)

Solution. Due to the $(a-b)^2 = a^2 - 2ab + b^2$ identity, it is sufficient to find the maximum of the sum

$$\sum_{k=1}^{n} \big(\sigma(k+1) - \sigma(k)\big)^2$$

where $\sigma(k)$ denotes the number from the k^{th} vertex and $\sigma(n+1) = \sigma(1)$. We will give an inductive algorithm to find an optimal arrangement and so we can find the maximal sum (or the minimal for the initial problem). Suppose we have an arbitrary arrangement with n numbers and construct an arrangement with $n+2$ numbers in the following way:

- Find the maximum of $\big|\sigma(k+1) - \sigma(k)\big|$. For such a k, denote $x = \min\big\{\sigma(k+1), \sigma(k)\big\}$ and $y = \max\big\{\sigma(k+1), \sigma(k)\big\}$.
- Increase each number by 1.
- Insert the numbers 1 and $n+2$ as in figure 1.

If we denote by s_{n+2} and s_n the corresponding distance sums, we have:

$$\begin{aligned} s_{n+2} &= s_n - (x-y)^2 + \big((n+1) - x\big)^2 + (n+1)^2 + y^2 \\ &= s_n + 2(n+1)^2 + 2xy - 2x - 2nx. \end{aligned}$$

On the other hand, from the obvious inequalities $x \geq 1$ and $n + 1 - y \geq 1$, we have $x(n+1-y) \geq 1$ and this implies $2xy - 2x - 2nx \leq -2$. Hence

$$s_{n+2} = s_n + 2(n+1)^2 - 2n = 2n(n+2).$$

If y_n is the maximal sum, we have $y_{n+2} = y_n + 2n(n+2)$ (because for $n = 3$ in the maximal arrangement $x = 1$, $y = 3$ and in each step the maximal distance $\big|\sigma(k+1) - \sigma(k)\big|$ occurs at $x = 1$ and $y = n$). For $n = 2$ and $n = 3$, we have $y_2 = 2$ and $y_3 = 6$ so from the obtained recurrence relation we can deduce $y_{2n} = 2 + \frac{8}{3}(n-1)n(n+1)$ and thus

$$x_{2n} = \frac{2\sum_{k=1}^{n} k^2 - y_{2n}}{2} = \frac{4n^3 + 6n^2 + 5n - 3}{3}$$

where x_n denotes the minimal sum for the initial problem. Analogously we have

$$x_{2n+1} = \frac{4n^3 + 12n^2 + 14n + 3}{3}.$$

For $n \in \{6, 7, 8, 9, 10\}$, we have illustrated the optimal arrangements on figures 2, 3, 4, 5 (in the exterior we have written the arrangement's numbers, inside the circle the product of any two adjacent number and in the inside circle the sum of these products).

Remark. For $p > 1$, the above arrangements will give the maximum of the sum $\sum_{k=1}^{n}\big(\sigma(k+1) - \sigma(k)\big)^p$, and this can be proved by the same method using the inequality

$$(n+1)^p + (n+1-x)^p + y^p - (y-x)^p \leq n^p + (n+1)^p + n^p - (n-1)^p.$$

□

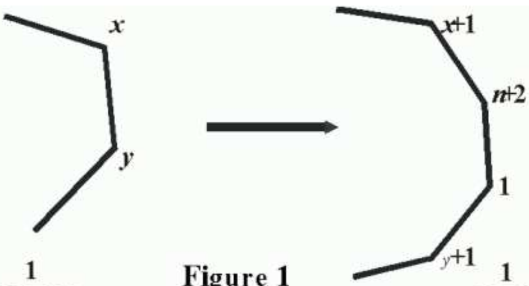

Figure 1

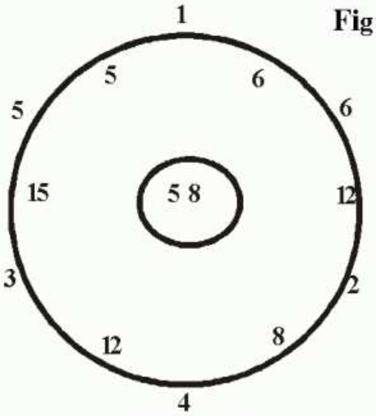

Figure 2

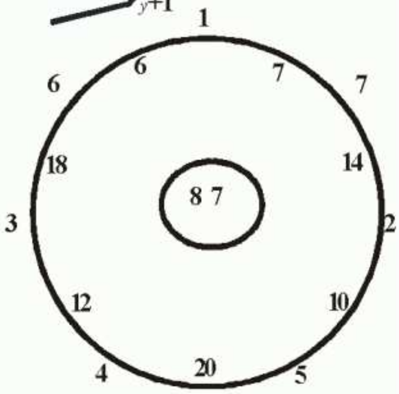

Figure 3

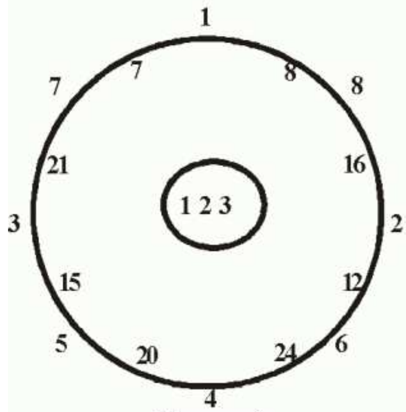

Figure 4

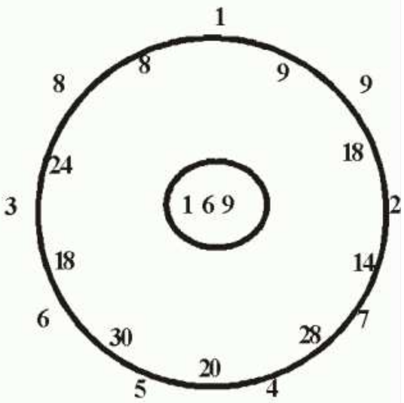

Figure 5

Problem j12-II-1/j12-II-56. *Find all complex solutions of the system*

$$\begin{aligned}(a+\mathrm{i}c)^3+(\mathrm{i}a+b)^3+(-b+\mathrm{i}c)^3&=-6\,,\\(a+\mathrm{i}c)^2+(\mathrm{i}a+b)^2+(-b+\mathrm{i}c)^2&=6\,,\\(1+\mathrm{i})a+2\mathrm{i}c&=0\,.\end{aligned}$$

(P. J. Šafárik University in Košice)

Solution. Let us notice that the third equation can be written as

$$(a+\mathrm{i}c)+(\mathrm{i}a+b)+(-b+\mathrm{i}c)=0;$$

that is why a natural substitution is

$$x=a+\mathrm{i}c,\qquad y=\mathrm{i}a+b,\qquad z=-b+\mathrm{i}c.$$

Then, our system is

$$\begin{aligned}x^3+y^3+z^3&=-6\\x^2+y^2+z^2&=6\\x+y+z&=0\end{aligned}$$

Using symmetric polynomials, we get

$$\begin{aligned}x+y+z&=\sigma_1,\\x^2+y^2+z^2&=(x+y+z)^2-2(xy+yz+xz)=\sigma_1^2-2\sigma_2,\\x^3+y^3+z^3&=(x+y+z)^3-3(xy+yz+xz)(x+y+z)+3xyz=\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3.\end{aligned}$$

It is a well-known fact that x, y, z must be roots of the cubic polynomial

$$f(t)=t^3-\sigma_1t^2+\sigma_2t-\sigma_3.$$

Since $\sigma_1=0$, $\sigma_1^2-2\sigma_2=6$, $\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3=-6$, we have

$$\sigma_1=0,\qquad\sigma_2=-3,\qquad\sigma_3=-2.$$

Rational roots of the polynomial $f(t)=t^3-3t+2$ can only be from the set $\{-2,-1,1,2\}$. Trying these, it turns out that $t=1$ and $t=-2$ are roots. Decomposition of the polynomial then reveals that 1 is a double root.

Thus, we have

$$(x,y,z)\in\big\{(1,1,-2),(1,-2,1),(-2,1,1)\big\}.$$

Returning back, we solve the system

$$\begin{aligned}a\qquad+\mathrm{i}c&=x\\\mathrm{i}a+b\qquad&=y\\-b+\mathrm{i}c&=z\end{aligned}$$

Its determinant is

$$|A|=\begin{vmatrix}1&0&\mathrm{i}\\\mathrm{i}&1&0\\0&-1&\mathrm{i}\end{vmatrix}=\mathrm{i}+1\neq0,$$

so for each (x,y,z) there is exactly one solution. It is easy to get the inverse matrix:

$$A^{-1}=\frac{1}{2}\begin{pmatrix}1+\mathrm{i}&-1-\mathrm{i}&-1-\mathrm{i}\\1-\mathrm{i}&1+\mathrm{i}&-1+\mathrm{i}\\-1-\mathrm{i}&1-\mathrm{i}&1-\mathrm{i}\end{pmatrix}.$$

Multiplying this matrix by the vectors (x,y,z) gives three solutions (a,b,c):

$$(1+\mathrm{i},2-\mathrm{i},-1),\qquad(1+\mathrm{i},-1-\mathrm{i},-1),\qquad(-2-2\mathrm{i},-1+2\mathrm{i},2).$$

One can easily verify that all three satisfy the system. □

Problem j12-II-2/j12-II-52. *A ring R (not necessarily commutative) contains at least one zero divisor and the number of zero divisors is finite. Prove that R is finite.*

(Eötvös Loránd University, Budapest)

Solution. Let m be the number of zero divisors and $u, v \in R$ two non-zero elements such that $uv = 0$.

We generate more zero divisors in the following way. For an arbitrary $x \in R$, the element xu is either 0 or also a zero divisor, since $(xu)v = x(uv) = 0$.†

If $xu = yu$ for some different elements $x, y \in R$, then $(x - y)u = 0$, and $x - y$ is a zero divisor. This implies that 0 or an arbitrary zero divisor can be obtained at most $m+1$ times in the form xu.‡

Thus, each of 0 and the m zero divisors is obtained at most m times and the number of elements of R cannot exceed $(m+1)^2$. □

† The set $\{xu; x \in R\}$ is finite, its cardinality being $\leq m + 1$.

‡ Define an equivalence relation: $x \sim y$ iff $xu = yu$. In each class of equivalence, there are $(m + 1)$ elements at most. Finally, the number of the classes of equivalence is equal to the cardinality of the set $\{xu; x \in R\}$, which is finite.

Problem j12-II-3/j12-II-53. *Let E be the set of all continuous functions $u\colon [0,1] \to \mathbb{R}$ satisfying*

$$u^2(t) \le 1 + 4\int_0^t s\big|u(s)\big|\,\mathrm{d}s, \quad \forall t \in [0,1].$$

Let $\varphi\colon E \to \mathbb{R}$ be defined by

$$\varphi(u) = \int_0^1 \big(u^2(x) - u(x)\big)\,\mathrm{d}x.$$

Prove that φ has a maximum value and find it. (Babeş-Bolyai University, Cluj-Napoca)

Solution. Let

$$v(t) = 1 + 4\int_0^t s\big|u(s)\big|\,\mathrm{d}s, \quad \forall t \in [0,1].$$

We have

$$v'(t) = 4t\big|u(t)\big| \le 4t\sqrt{1 + 4\int_0^t s\big|u(s)\big|\,\mathrm{d}s} \le 4t\sqrt{v(t)}$$

so

$$\sqrt{v(t)} - 1 = \int_0^t \frac{v'(s)}{2\sqrt{v(s)}}\,\mathrm{d}s \le \int_0^t 2s\,\mathrm{d}s = t^2$$

therefore

$$\big|u(t)\big| \le \sqrt{v(t)} \le t^2 + 1.$$

If we consider φ, we have

$$\big|u^2(t) - u(t)\big| = \big|u(t)\big|\big|u(t) - 1\big| \le (t^2+1)(t^2+2),$$
$$\big|\varphi(u)\big| \le \int_0^1 \big|u^2(t) - u(t)\big|\,\mathrm{d}t \le \int_0^1 (t^2+1)(t^2+2)\,\mathrm{d}t = \frac{16}{5}.$$

Equality can be achieved if

$$\big|u(t)\big| = t^2 + 1 \qquad \text{and} \qquad \big|u(t) - 1\big| = t^2 + 2.$$

This is the case of $u(t) = -t^2 - 1$, which belongs to E. □

Problem j12-II-4/j12-II-62. *Prove that*

$$\lim_{n\to\infty} n^2\left(\int_0^1 \sqrt[n]{1+x^n}\,\mathrm{d}x - 1\right) = \frac{\pi^2}{12}.$$

(Sofia University St. Kliment Ohridski)

Solution. We will prove that

$$\lim_{n\to\infty} n^2\left(\int_0^1 \sqrt[n]{1+x^n}\,\mathrm{d}x - 1\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = \frac{\pi^2}{12}.$$

Let $a_n = n^2\big(\int_0^1 \sqrt[n]{1+x^n}\,\mathrm{d}x - 1\big)$. It is widely known that $(1+t)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}t^k$ for any $t\in[0,1]$ and $\alpha\in(0,1)$. Moreover, $\big|\binom{\alpha}{k}t^k\big| = (-1)^{k-1}\binom{\alpha}{k}t^k$ and $\big|\binom{\alpha}{k}t^k\big| \geq \big|\binom{\alpha}{k}t^{k+1}\big|$ for $k\geq 1$, $t\geq 0$ and $\alpha\in(0,1)$. Thus, the following inequalities hold:

$$\sum_{k=0}^{2p}\binom{\alpha}{k}t^k \leq (1+t)^\alpha \leq \sum_{k=0}^{2p+1}\binom{\alpha}{k}t^k.$$

Let us put $t = x^n$ and $\alpha = \frac{1}{n}$. Integrating on $[0,1]$, we obtain

$$\sum_{k=0}^{2p}\binom{1/n}{k}\frac{1}{nk+1} \leq \int_0^1 \sqrt[n]{1+x^n}\,\mathrm{d}x \leq \sum_{k=0}^{2p+1}\binom{1/n}{k}\frac{1}{nk+1}.$$

Hence,

$$0 \leq a_n - n^2\sum_{k=1}^{2p}\binom{1/n}{k}\frac{1}{nk+1} \leq n^2\binom{1/n}{2p+1}\frac{1}{n(2p+1)+1}.$$

A simple calculation gives the following estimation:

$$n^2\binom{1/n}{2p+1}\frac{1}{n(2p+1)+1} \leq \frac{1}{(2p+1)^2}.$$

Consequently, as n tends to infinity,

$$0 \leq \limsup_{n\to\infty}\left(a_n - \sum_{k=1}^{2p}\frac{(-1)^k}{k^2}\right) \leq \frac{1}{(2p+1)^2}$$

and

$$0 \leq \liminf_{n\to\infty}\left(a_n - \sum_{k=1}^{2p}\frac{(-1)^k}{k^2}\right) \leq \frac{1}{(2p+1)^2}.$$

Letting $p\to\infty$, we obtain the desired result. □