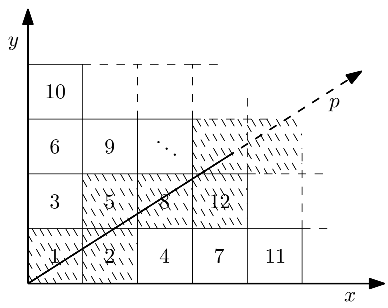
The 10$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 5$^{\text{th}}$ April 2000
Category I

Problem 1 *Is there a countable set Y and an uncountable family $\mathcal{F}$ of its subsets, such that for every two distinct $A, B \in \mathcal{F}$ their intersection $A \cap B$ is finite?*

Solution The answer is yes.

Put all natural numbers $\mathbb{N}$ in a coordinate system like in the picture.



For each ray p we put in the set A_p all numbers, which p intersects (intersects its square), and we place all such sets A_p into $\mathcal{F}$. For each ray p we assign the angle α_p, which is between p and x. Since all α_p form the interval $(0, \frac{\pi}{2})$, the set of rays p is uncountable. Furthermore, for different rays p_1 and p_2, the intersection $A_{p_1} \cap A_{p_2}$ is finite, because there exists distance $d_{p_1 p_2}$ from the origin of the coordinate system where rays are far enough, so they will not pass through the same square any more. The problem is solved. $\square$

Solution Yes. Let $Y = \mathbb{N}$ and denote S set of all infinite sequences $\{a_n\}$ of 0 and 1. To each sequence $\{a_n\}$ assign the set $C_{\{a_n\}} \in \mathcal{F}$ in the following way:

$$C_{\{a_n\}} = \left\{ 2^k + \sum_{n=1}^{k} a_n \cdot 2^{n-1}, \text{ for } k = 1, 2, \dots \right\}.$$

Suppose we have two distinct sets $C_{\{a_n\}}$, $C_{\{b_n\}}$ for some distinct $\{a_n\}, \{b_n\} \in \mathcal{F}$.

Suppose now

$$2^{k_1} + \sum_{n=1}^{k_1} a_n \cdot 2^{n-1} = 2^{k_2} + \sum_{n=1}^{k_2} b_n \cdot 2^{n-1} \text{ for some } k_1,\ k_2 \in \mathbb{N},$$

$$\text{where } \left(2^{k_1} + \sum_{n=1}^{k_1} a_n \cdot 2^{n-1} \right) \in C_{\{a_n\}} \text{ and } \left(2^{k_2} + \sum_{n=1}^{k_2} b_n \cdot 2^{n-1} \right) \in C_{\{b_n\}}.$$

This is like equality of two numbers written in binary system. For the number

$$2^{k_1} + \sum_{n=1}^{k_1} a_n \cdot 2^{n-1}$$

is the first digit regarding to the 2^{k_1}th digit 1, and the next digits are $a_{k_1-1}, a_{k_1-2}, \dots, a_1$. Analogous are the digits for the other number. So these numbers are equal if and only if $k_1 = k_2$ and $a_i = b_i$ for all $i < k_1$. So if the sets $C_{\{a_n\}}$, $C_{\{b_n\}}$ contain infinitely many of the same numbers then $a_i = b_i$ for all $i \in \mathbb{N}$ and $C_{\{a_n\}} = C_{\{b_n\}}$, a contradiction. $\square$

The 10^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 5^{th} April 2000
Category I

Problem 2 *Let* $f\colon \mathbb{N} \to \mathbb{R}$ *be given by*

$$f(n) = n^{\frac{\tau(n)}{2}},$$

$n \in \mathbb{N} = \{1, 2, \dots\}$, $\tau(n)$ *- the number of divisors of* n*. Show that* f *is injective into* $\mathbb{N}$.

Solution Recall that every natural number n greater than 1 can be written uniqely (up to order) as a product

$$n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k},$$

where the p_i are different primes and the a_i are natural numbers. Then

$$\tau(n) = (1 + a_1)(1 + a_2) \dots (1 + a_k).$$

Now we show that for every natural number n, $f(n)$ is also a natural number:

- $f(1) = 1$ is natural,
- if $n > 1$ and $\tau(n)$ is even, then $\frac{\tau(n)}{2}$ is natural and so is $f(n)$,
- if $n > 1$ and $\tau(n)$ is odd, then all a_i are even so n is a square of a natural number and $f(n)$ is natural too.

Now it is easy to see that prime number p divides the natural number n if and only if it divides $f(n)$. We use this fact to prove injectivity of f.

Let $f(m) = f(n)$. Then m and n are divisible by the same primes and we can write $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, $m = p_1^{b_1} p_2^{b_2} \dots p_k^{b_k}$. As $f(n) = f(m)$, it is true that $a_i \tau(n) = b_i \tau(m)$, $i = 1, 2, \dots k$. We can assume that $\tau(n) \le \tau(m)$ (if not we change m and n). Using this we get $a_i \ge b_i$ and $\tau(n) \ge \tau(m)$. From this follows that $\tau(n) = \tau(m)$, $a_i = b_i$ and $n = m$. $\square$

Solution Let us write n in the form $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p_1, p_2, ..., p_k are prime divisors (this representation is unique), and let $1 = d_1 < d_2 < \dots < d_{\tau(n)} = n$ be all its divisors. Then

$$d_1 d_2 \dots d_{\tau(n)} = \sqrt{(d_1 d_{\tau(n)}) \cdot (d_2 d_{\tau(n)-1}) \dots (d_{\tau(n)} d_1)} = \sqrt{n^{\tau(n)}} = n^{\frac{\tau(n)}{2}} = f(n),$$

since $d_k d_{\tau(n)-k+1}$=n. So $f(n)$ is natural, because it can be expressed as multiple of natural numbers d_1, ..., $d_{\tau(n)}$. Suppose now $f(n) = f(m)$ for distinct natural numbers m, n. From $m^{\frac{\tau(m)}{2}} = f(m) = f(n) = n^{\frac{\tau(n)}{2}}$ it follows that $m = n^{\frac{\tau(n)}{\tau(m)}}$. This implies that m and n have the same set of prime divisors, so m can be writen as $m = p_1^{\beta_1} p_2^{\beta_2} \dots p_k^{\beta_k}$. The obtained relation $m = n^{\frac{\tau(n)}{\tau(m)}}$ implies $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \dots = \frac{\alpha_k}{\beta_k} = c$. Without loss of generality we can suppose that $m > n$. Then $c > 1$, and, since all α_i and β_i are positive integers, m is divisible by n. So the set of all divisors of n is a subset of the set of all divisors of m, and $\tau(m) \ge \tau(n)$. From $m > n$ we can conclude that $m^{\frac{\tau(m)}{2}} > n^{\frac{\tau(n)}{2}}$, a contradiction. $\square$

Problem 3 *Let $a_1, a_2, \dots$ be a bounded sequence of reals. Is it true that*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} a_n = b \quad \textit{and} \quad \lim_{N\to\infty} \frac{1}{\log N}\sum_{n=1}^{N} \frac{a_n}{n} = c$$

imply $b = c$?

Solution We prove that if $\frac{1}{N}\sum\limits_{n=1}^{N} a_n \to b$ for any (not even necessarily bounded) sequence of reals then

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{a_n}{n} \to b\,.$$

Assume that

$$\frac{1}{N}\sum_{n=1}^{N} a_n \to b\,.$$

Define

$$h_N = \sum_{n=1}^{N} a_n\,.$$

(We have $h_0 = 0$.)

Then by our assumption we have $h_N \to b$ and by definition we get

$$a_N = Nh_N - (N-1)h_{N-1} = N(h_N - h_{N-1}) + h_{N-1}\,.$$

Thus

$$\sum_{n=1}^{N}\frac{a_n}{n} = \sum_{n=1}^{N} h_n - h_{n+1} + \frac{h_{n-1}}{n} = h_N + \sum_{n=1}^{N}\frac{h_{n-1}}{n}\,.$$

Therefore

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{a_n}{n} = \frac{h_N}{\log N} + \sum_{n=1}^{N}\frac{h_n}{n}\,.$$

Since h_N converges the first term goes to 0. As $\frac{1}{\log N}\sum\limits_{n=1}^{N}\frac{1}{n} \to 1$ and $h_N \to b$ we get that $\frac{1}{\log N}\sum\limits_{n=1}^{N}\frac{a_n}{n} \to b$. ☐

Solution Denote $\frac{1}{n}\sum\limits_{i=1}^{n} a_i = b_n$, $n \geq 1$, $b_0 = 0$ and $\frac{1}{\log n}\sum\limits_{i=1}^{n}\frac{a_i}{i} = c_n$, $n \geq 1$. The sequence $\{b_n\}_{n\in\infty}$ converges to b.

Since $a_n = nb_n - (n-1)b_{n-1}$, we obtain

$$\begin{aligned} c_n &= \frac{1}{\log n}\sum_{i=1}^{n}\frac{a_i}{i} = \frac{1}{\log n}\sum_{i=1}^{n}\frac{ib_i - (i-1)b_{i-1}}{i} \\ &= \frac{1}{\log n}\sum_{i=1}^{n}\big(b_i - \frac{i-1}{i}b_{i-1}\big) = \frac{1}{\log n}\big(b_n - \sum_{i=1}^{n-1}\frac{b_i}{i+1}\big). \end{aligned}$$

Let us write $b_n = b + \varepsilon_n$, where $\varepsilon \to 0$. Then

$$c_n = \frac{1}{\log n}\left(\Big(\sum_{i=1}^{n}\frac{1}{i}\Big)b + \sum_{i=1}^{n-1}\frac{\varepsilon_i}{i+1} + \varepsilon_n\right) = \frac{\sum_{i=1}^{n}}{\log n}b + \frac{\sum_{i=1}^{n-1}\frac{\varepsilon_i}{i+1}}{\log n} + \frac{\varepsilon_n}{\log n}.$$

Easily we see $\frac{\sum_{i=1}^{n}\frac{1}{i}}{\log n} \to 1$, $\frac{\varepsilon_n}{\log n} \to 0$. For the rest we have

$$\frac{\sum_{i=1}^{n-1}\frac{\varepsilon_i}{i+1}}{\log n} = \frac{\sum_{i=1}^{k}\frac{\varepsilon_i}{i+1}}{\log n} + \frac{\sum_{i=k+1}^{n-1}\frac{\varepsilon_i}{i+1}}{\log n}$$

and it follows that

$$\frac{\sum_{i=1}^{k} \frac{\varepsilon_i}{i+1}}{\log n} \to 0 \quad \text{for } n \to \infty,$$

$$\left| \frac{\sum_{i=k+1}^{n-1} \frac{\varepsilon_i}{i+1}}{\log n} \right| \le \sup_{j \ge k}\{|\varepsilon_j|\} \cdot \frac{\sum_{i=k+1}^{n-1} \frac{1}{i}}{\log n} \le \sup_{j \ge k}\{|\varepsilon_j|\} \quad \text{for all } n \ge k.$$

Since $\sup_{j \ge k}\{|\varepsilon_j|\} \to 0$ for $k \to \infty$ and our choice of k can be arbitrary large, we got $c_n \to b$, and the problem is solved. □

The 10^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 5^{th} April 2000
Category I

Problem 4 *Let us choose arbitrarily n vertices of a regular $2n$-gon and colour them red. Remaining vertices are coloured blue. We arrange all red-red distances into a nondecreasing sequence and do the same with blue-blue distances. Prove that the sequences are equal.*

Solution Let $1 \geq m \geq n$ be the combinatorial distance (i.e. the Euclidean distance d_m between the first and $(m+1)$-st vertices corresponds to m). For a given m denote by k the gcd of $2n$ and m. The original $2n$-gon can be decomposed into k disjoint $\frac{2n}{k}$-gons of edge lenght d_m. Assume that there are r_i red and b_i blue vertices on the i-th $\frac{2n}{k}$-gon, and denote c_i to be the number of oriented red $\rightarrow$ blue changes between the neighboring vertices. The number of neighboring blue pairs equals $r_i - c_i$, while the number of neighboring blue pairs equals $b_i - c_i$. The total number of neighoring red pairs will be $\sum_i (r_i - c_i) = n - \sum_i c_i$ which is the same as the total number $\sum_i (b_i - c_i) = n - \sum_i c_i$ of neighboring blue pairs. $\square$

Solution It is enough to show, that the number of each particular distance is same among both red–red and blue–blue pairs. Choose one of the used distances, let us say d. Denote d_{rr} to be the number of red–red pairs with distance d and analogously d_{bb} number of blue–blue pairs and d_{rb} number of red–blue pairs with distance d. Clearly $2d_{rr} + d_{rb} = 2n$, since $2\times$ number of red–red pairs plus $1\times$ red–blue pairs gives $2\times$ number of red vertices. Analogously $2d_{bb} + d_{rb} = 2n$. By subtracting one relation from the other we get $2d_{bb} = 2d_{rr}$, which is the end. $\square$

The 10^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 5^{th} April 2000
Category II

Problem 1 *Let p be a prime of the form $p = 4n - 1$ where n is a positive integer. Prove that*

$$\prod_{k=1}^{p}(k^2+1) \equiv 4 \pmod{p}.$$

Solution Consider the polynomials $P(x) = \prod_{k=1}^{p}(k^2 - x^2)$ and $p(x) = \prod_{k=1}^{p-1}(k - x)$. Easily we see that $P(x) = p(x) \cdot p(-x) \cdot (p^2 - x^2)$.

Since $p(x)$ is of degree $p-1$ and has roots $1, 2, \ldots, p-1$, $p(x)$ is congruent modulo p to the polynomial $q(x) = x^{p-1} - 1$, which has also roots $1, 2, \ldots, p-1$. Therefore

$$P(x) = p(x)p(-x)(p^2 - x^2) \equiv q(x)q(-x)(p^2 - x^2) \equiv q(x)q(-x)(-x^2) \pmod{p}$$

and it follows that

$$\begin{aligned}\prod_{k=1}^{p}(k^2+1) = P(i) &\equiv q(i) \cdot q(-i) \cdot (-i^2) = (i^{p-1} - 1) \cdot \big((-i)^{p-1} - 1\big) \\ &= (i^{4n-2} - 1) \cdot \big((-i)^{4n-2} - 1\big) = (-2) \cdot (-2) = 4 \pmod{p}.\end{aligned}$$

$\square$

The 10^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 5^{th} April 2000
Category II

Problem 2 *If we write the sequence AAABABBB along the perimeter of a circle, then every word of the length 3 consisting of letters A and B (i.e. AAA, AAB, ABA, BAB, ABB, BBB, BBA, BAA) occurs exactly once on the perimeter. Decide whether it is possible to write a sequence of letters from a k-element alphabet along the perimeter of a circle in such a way that every word of the length l (i.e. an ordered l-tuple of letters) occurs exactly once on the perimeter.*

Solution Let us denote the alphabet by P. Let us form the directed graph $G = (V, E)$, where $V = \{[a_1, \ldots, a_{l-1}]; a_i \in P\}$ and $E = \{\big[[a_1, \ldots, a_{l-1}], [b_1, \ldots, b_{l-1}]\big]; a_2 = b_1, a_3 = b_2, \ldots, a_{l-1} = b_{l-2}\}$.
First, considering any two vertices $[a_1, \ldots, a_{l-1}]$ and $[b_1, \ldots, b_{l-1}]$, we find that there must be at least one oriented path between them:

$$[a_1, \ldots, a_{l-1}] \to [a_2, \ldots, a_{l-1}, b_1] \to [a_3, \ldots, a_{l-1}, b_1, b_2] \to \cdots \to [b_1, \ldots, b_{l-1}]$$

(some vertices and arcs can repeat in the sequence). This implies that the graph is strongly connected.
Second, we realize that every vertex $[a_1, \ldots, a_{l-1}]$ has exactly k outgoing and k ingoing arcs: the outgoing arcs are directed to the vertices $[a_2, \ldots, a_{l-1}, o]$, where o goes through the the whole alphabet P, and the ingoing come from vertices $[i, a_1, \ldots, a_{l-2}]$, where i also goes through the whole alphabet. That means that the inner and outer degrees of every vertex are identical and the graph is an Euler (directed) graph. As a consequence, there exists an Eulerian cycle in it, i.e. a cycle containing all arcs going through every arc exactly once. We can now form the searched for cyclic sequence as follows:
Let us start with an arbitrary vertex and write down its sequence $a_1, \ldots, a_{l-1}$.
Let us follow the Eulerian cycle and add the last letter of every vertex to the sequence until we reach the starting vertex again. Now we delete the last $l-1$ letters (which are necessarily the same as the starting ones).
Since there is a bijection between the set of all l-letter words and the set of arcs V:

$$[a_1, \ldots, a_l] \longleftrightarrow \big[[a_1, \ldots, a_{l-1}], [a_2, \ldots, a_l]\big],$$

it is clear that the sequence has the demanded properties. □

The 10$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 5$^{\text{th}}$ April 2000
Category II

Problem 3 *Let m, n be positive integers and let $x \in [0,1]$. Prove that*

$$(1-x^n)^m + \big(1-(1-x)^m\big)^n \geq 1\,.$$

Solution We will prove that $(1-x^n)^m \geq 1-(1-(1-x)^m)^n$.

Take an $m \times n$ chessboard. The probability, that one particular square is black, is $x \in [0,1]$, the probability of being white is $x-1$. Assume this for all squares. Then

$(1-x)^m$ is the probability that the whole row is white

$1-(1-x)^m$ is the probability that there is at least one black in the row

$(1-(1-x)^m)^n$ is the probability that in each row there is at least one black

$1-(1-(1-x)^m)^n$ is the probability that at least one row does not contain a black.

Denote by A the last event, in which some row does not a contain black (it is all white). We continue:

$1-x^n$ is the probability that the column contains at least one white

$(1-x^n)^m$ is the probability that each column contains at least one white.

Denote by B the event in which each column contains at least one white. It is clear that $A \subset B$, because if one row is white then each column contains some white. Therefore is $P(b) \geq P(A)$, written in the other form: $(1-x^n)^m \geq 1-(1-(1-x)^m)^n$. $\square$

The 10^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 5^{th} April 2000
Category II

Problem 4 *Let $\mathcal{B}$ be a family of open balls in $\mathbb{R}^n$ and $c < \lambda\big(\bigcup \mathcal{B}\big)$ where λ is the n-dimensional Lebesgue measure. Show that there exists a finite family of pairwise disjoint balls $\{U_i\}_{i=1}^k \subseteq \mathcal{B}$ such that*

$$\sum_{j=1}^{k} \lambda(U_j) > \frac{c}{3^n} .$$

Solution Suppose first $\sup\limits_{U\in\mathcal{B}} \lambda(U) < \infty$ for $U \in \mathcal{B}$. In other case we just take a large enough ball U_0 for which $\lambda(U_0) > \frac{c}{3^n}$.

Take $\varepsilon > 0$. We first construct the infinite sequence $\{U_k\}_{k=1}^{\infty}$ of disjoint balls for which $\lambda\Big(\bigcup U_k\Big) \geq \frac{\lambda(\bigcup \mathcal{B})}{(3+\varepsilon)^n}$. The procedure is the following. In the k-th step count $\sup \lambda(U)$ through all the rest of the balls in $\mathcal{B}$, then choose U_k such that $\lambda(U_k) \geq \sup \lambda(U) \cdot \big(\frac{3}{3+\varepsilon}\big)^n$ and remove from $\mathcal{B}$ all balls U which intersect ball U_k. Continue to infinity by the $(k+1)$-th step. It is clear, that the balls in the constructed sequence $\{U_n\}$ are disjoint. Further, if we increase each ball U_k from $\{U_k\}_{k=1}^{\infty}$ $(3+\varepsilon)$-times, than they will contain all of the set $\mathcal{B}$. This holds, because ball U_i increased $(3+\varepsilon)$-times covers all balls intersecting U_i removed from $\mathcal{B}$ in i-th step. It follows that $\lambda\Big(\bigcup U_k\Big) \geq \frac{\lambda(\bigcup \mathcal{B})}{(3+\varepsilon)^n}$. If $\lambda\big(\bigcup \mathcal{B}\big) < \infty$, then there is $k_0 \in \mathbb{N}$ for which

$$\lambda\Bigg(\bigcup_{k=1}^{k_0} U_k\Bigg) \geq \frac{\lambda\big(\bigcup \mathcal{B}\big)}{(3+2\varepsilon)^n} .$$

By choosing ε so that $\frac{\lambda(\bigcup \mathcal{B})}{(3+2\varepsilon)^n} > \frac{c}{3^n}$ (this is always possible) we solve the problem. In case $\lambda\big(\bigcup \mathcal{B}\big) = \infty$ we just choose k_0 big enough to exceed the finite constant c in the desired inequlity. $\square$