The 7th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 9th April 1997
Category I

Problem 1. Let a be an odd positive integer. Prove that if d divides (a^2+2), then $d \equiv 1 \pmod 8$ or $d \equiv 3 \pmod 8$. [10 points]

Problem 2. Let $\alpha \in (0,1]$ be a given real number and let a real sequence $\{a_n\}_{n=1}^{\infty}$ satisfy the inequality

$$a_{n+1} \leq \alpha a_n + (1-\alpha)a_{n-1} \qquad \text{for } n = 2,\, 3,\, \ldots$$

Prove that if $\{a_n\}$ is bounded, then it must be convergent.
[12 points]

Problem 3. Let $c_1, c_2, \ldots, c_n$ be real numbers such that

$$c_1^k + c_2^k + \cdots + c_n^k > 0 \qquad \text{for all } k = 1,\, 2,\, \ldots$$

Let us put

$$f(x) = \frac{1}{(1-c_1x)(1-c_2x)\ldots(1-c_nx)}\,.$$

Show that $f^{(k)}(0) > 0$ for all $k = 1, 2, \ldots$ [15 points]

Problem 4-M. Find all real numbers $a > 0$ for which the series

$$\sum_{n=1}^{\infty} \frac{a^{f(n)}}{n^2}$$

is convergent; $f(n)$ denotes the number of 0's in the decimal expansion of n. [13 points]

Problem 4-I. Let us declare

```
const N_MAX = 255;
type tR = array [1..N_MAX] of real;
     tN = array [1..N_MAX] of integer;
```

and let random be a function with no arguments which returns real random values distributed uniformly in $[0,1)$.

You are to choose K unique random integer numbers ($1 \leq K \leq N \leq N\_MAX$) from 1 to N so that the probability of the choice of a number i is equal to a given P_i (if the number has not been chosen yet; the choice is to be repeated otherwise), $\sum_{i=1}^{N} P_i = 1$.

Write a procedure in Pascal that returns such K integer numbers in the first K elements of the vector of the type tN. The input arguments of the procedure are K, N and the vector of P_i's. [13 points]