The 6th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 3rd April 1996
Category I

Problem 1 *On the ellipse* $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$ *find the point* $T = [x_0, z_0]$ *such that the triangle bounded by the axes of the ellipse and the tangent at that point has the least area.*

Solution The equation of the tangent at point T is

$$\frac{xx_0}{a^2} + \frac{zz_0}{b^2} = 1$$

and its intersection with the axes occurs at the points $P = [\frac{a^2}{x_0}, 0]$ and $Q = [0, \frac{b^2}{z_0}]$. The area S of triangle OPQ is

$$S = \frac{b^2a^2}{2z_0x_0} = \frac{b^3a}{2z_0\sqrt{b^2 - z_0^2}}\,.$$

We will find the extreme point of the function $S(z_0)$. Note that

$$S'(z_0) = \frac{b^3a}{2}\Big(\frac{1}{(b^2 - z_0^2)^{\frac{3}{2}}} - \frac{1}{z_0^2\sqrt{b^2 - z_0^2}}\Big)\,.$$

Hence $z_0 = \frac{b}{\sqrt{2}}$ and $x_0 = \frac{a}{\sqrt{2}}$. At the points $z_0 = b$ and $z_0 = 0$, where the derivative does not exist, the function $S(z_0)$ does not have an extremum, because we do not have a triangle. And because $S''(\frac{b}{\sqrt{2}}) > 0$ we have a minimum at that point. If we use the same process to find the additional points, we get the other three points $[-\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}]$, $[\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}]$ and $[-\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}]$. □

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Problem 2 *Let* $\{a_n\}_{n=0}^{\infty}$ *be the sequence of integers such that* $a_0 = 1, a_1 = 1, a_{n+2} = 2a_{n+1} - 2a_n$. *Decide weather*

$$a_n = \sum_{k=0}^{[n/2]} \binom{n}{2k}.$$

Solution No. If

$$a_n = \sum_{k=0}^{[n/2]} \binom{n}{2k},$$

then we get the sequence $a_n = 2^{n-1}$ for $n \geq 1$. But if $a_{n+2} = 2a_{n+1} - 2a_n$ holds, then we have the another sequence $a_n = \frac{1}{2}(1+\mathrm{i})^n + \frac{1}{2}(1-\mathrm{i})^n$ for $n \geq 0$. □

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Problem 3 *Prove that the equation*

$$\frac{z}{1+z^2}+\frac{y}{1+y^2}+\frac{x}{1+x^2}=\frac{1}{1996}$$

has finitely many solutions in positive integers.

Solution Let $x \le y \le z$. Then

$$\frac{z}{1+z^2} \le \frac{y}{1+y^2} \le \frac{x}{1+x^2}\,. \tag{1}$$

Hence

$$\frac{3x}{1+x^2} \ge \frac{1}{1996}$$

and from this $x^2-5988x+1 \le 0$ so we have finitely many $x \in \mathbb{N}$. Further we can write

$$\frac{z}{1+z^2}+\frac{y}{1+y^2}=\frac{1}{C_x}\,,$$

where $C_x = \frac{1}{1996}-\frac{x}{1+x^2}$ is bounded number (for particular x). From (1) we get

$$\frac{2y}{1+y^2} \ge \frac{1}{C_x}$$

and from this $y^2-2C_xy+1 \le 0$ and y is also a bounded number. Then

$$\frac{z}{1+z^2}=\frac{1}{1996}-\frac{x}{1+x^2}-\frac{y}{1+y^2}\,,$$

where the term on the right side has finitely many values. For each particular value of $\frac{1}{1996}-\frac{x}{1+x^2}-\frac{y}{1+y^2}$ we obtain two distinct (or the same) numbers z. So we have only finitely many numbers z.

Hence the equation

$$\frac{z}{1+z^2}+\frac{y}{1+y^2}+\frac{x}{1+x^2}=\frac{1}{1996}$$

has finitely many solutions. □

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Category II

Problem 1 *Is it possible to cover the plane with the interiors of a finite number of parabolas?*

Solution Suppose that there exists a finite system $\mathcal{S}$ of parabolas, which cover the plane. The number of parabolas is n. Take two parabolas from $\mathcal{S}$ which intersect. These parabolas have at most four intersection points. We choose another parabola from $\mathcal{S}$ which covers at least one of the intersection points. Hence we have either a new intersection point or part of a parabola which is not covered. We do this iteration process for all parabolas from $\mathcal{S}$. At the end of this iteration process we get one intersection point or part of a parabola which is not covered. So we have a contradiction. $\square$

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Problem 2 *Let* $\{x_n\}_{n=0}^{\infty}$ *be the sequence such that* $x_0 = 2, x_1 = 1$ *and* x_{n+2} *is the remainder of the number* $x_{n+1} + x_n$ *divided by* 7*. Prove that* x_n *is the remainder of the number*

$$4^n \sum_{k=0}^{[n/2]} 2\binom{n}{2k} 5^k$$

divided by 7*.*

Solution

□

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Problem 3 *Let* $\mathrm{cif}(x)$ *denote the sum of the digits of the number* x *in the decimal system. Put* $a_1 = 1997^{1996^{1997}}$, $a_{n+1} = \mathrm{cif}(a_n)$ *for every* $n > 0$. *Find* $\lim_{n\to\infty} a_n$.

Solution For the function $\mathrm{cif}(x)$ we have $\mathrm{cif}(x) \equiv x \pmod 9$. For $x \geq 10$ we obtain $\mathrm{cif}(x) < x$ and for $x \leq 9$ we obtain $\mathrm{cif}(x) = x$. Hence there exists N such that $1 \leq a_{n+1} < a_n$ for all $n \leq N-1$ and $1 \leq a_{n+1} = a_n$ for all $n \geq N$. This implies that

$$\lim_{n\to\infty} a_n = a_N \equiv a_1 \pmod 9 .$$

We have

$$1997^{1996^{1997}} \equiv (-1)^{1996^{1997}} = 1 \pmod 9 .$$

From $1 \leq \lim_{n\to\infty} a_n \leq 9$ we obtain $\lim_{n\to\infty} a_n = 1$. □