The 28th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13th April 2018
Category II

Problem 1 *Find all real solutions of the equation*

$$17^x + 2^x = 11^x + 2^{3x}\,.$$

[10 points]

Solution Let us rewrite the equation as

$$17^x - 11^x = 8^x - 2^x.$$

It's easy to see that $x = 0$ is a solution. Fix $x \in \mathbb{R} \setminus \{0\}$ and suppose that it is a solution to our problem. Consider the function $f(t) = t^x$. By the mean value theorem applied on the interval $[2, 8]$ there is $t_1 \in (2, 8)$ such that

$$6f'(t_1) = f(8) - f(2) = 8^x - 2^x.$$

Again – by the mean value theorem on $[11, 17]$ we get

$$6f'(t_2) = f(17) - f(11) = 17^x - 11^x.$$

Since x is a solution, we have

$$6f'(t_1) = 6f'(t_2).$$

Since $x \neq 0$ we have

$$6xt_1^{x-1} = 6xt_2^{x-1} \;\Rightarrow\; t_1^{x-1} = t_2^{x-1}$$

and $(t_1/t_2)^{x-1} = 1$. Therefore $x = 1$, since $t_1 < t_2$, and it's easy to check that it is also a solution. □

The 28th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13th April 2018
Category II

Problem 2 *Let n be a positive integer and let $a_1 \le a_2 \le \cdots \le a_n$ be real numbers such that*

$$a_1 + 2a_2 + \cdots + na_n = 0\,.$$

Prove that

$$a_1[x] + a_2[2x] + \cdots + a_n[nx] \ge 0$$

for every real number x. (Here $[t]$ denotes the integer satisfying $[t] \le t < [t]+1$.) [10 points]

Solution We proceed by induction on n. For $n = 1$ the condition forces $a_1 = 0$ and the statement becomes trivial.

Suppose that the statement is true for some n and take $n+1$ numbers $a_1 \le \ldots \le a_{n+1}$ statisfying the constraints. Notice that $a_{n+1} \ge 0$ due to the ordering.

For $i = 1, \ldots, n$, let $b_i = a_i + \dfrac{2a_{n+1}}{n}$. These numbers are ordered in increasing order, and

$$\sum_{i=1}^{n} ib_i = \sum_{i=1}^{n}\left(a_i + \frac{2a_{n+1}}{n}\right) = \sum_{i=1}^{n} ia_i + \frac{2}{n}(1+2+\ldots+n)a_{n+1} = \sum_{i=1}^{n+1} ia_i = 0.$$

By the induction hypothesis,

$$0 \le \sum_{i=1}^{n} b_i[ix] = \sum_{i=1}^{n} a_i[ix] + a_{n+1}\cdot\frac{2}{n}\sum_{i=1}^{n}[ix].$$

Applying $\big[ix\big] + \big[(n+1-i)x\big] \le \big[(n+1)x\big]$ in the last sum, we conclude that

$$\frac{2}{n}\sum_{i=1}^{n}[ix] \le \big[(n+1)x\big].$$

Due to $a_{n+1} \ge 0$, we get

$$0 \le \sum_{i=1}^{n} a_i[ix] + a_{n+1}\cdot\frac{2}{n}\sum_{i=1}^{n}[ix] \le \sum_{i=1}^{n+1} a_i[ix].$$

$\square$

Problem 3 *In $\mathbb{R}^3$ some n points are coloured. In every step, if four coloured points lie on the same line, Vojtěch can colour any other point on this line. He observes that he can colour any point $P \in \mathbb{R}^3$ in a finite number of steps (possibly depending on P). Find the minimal value of n for which this could happen.* [10 points]

Solution Answer: for $n = \binom{6}{3} = 20$.

Example for 20 points: take any 6 planes $\alpha_i, 1 \leqslant i \leqslant 6$, in general position and mark their triple intersections. Then all points on the lines $\alpha_i \cap \alpha_j$ may be marked. Any point P in the plane α_i may be marked too: draw a line through P in general position, it meets five lines $\alpha_i \cap \alpha_j, j \neq i$, in five points which may be marked. It remains do the same with arbitrary point P in the space: draw a line through P, which meets α_i's in six markable points.

Assume that $n \leqslant 19$. Then there exists a non-zero polynomial $p(x, y, z)$ of degree at most 3 such that $p(A) = 0$ for all marked points. Indeed, the space of such polynomials has dimension $20 > 19$. Note that this property hold true for all points which we may mark: if $p(A_i) = 0$ for different points A_1, A_2, A_3, A_4 on a line, then $p(x) = 0$ on the whole line cause of $\deg p \leqslant 3$. Therefore we can not mark a point B for which $p(B) \neq 0$. □

Problem 4 *Compute the integral*

$$\iint\limits_{\mathbb{R}^2} \left(\frac{1-\mathrm{e}^{-xy}}{xy}\right)^2 \mathrm{e}^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y\,.$$

[10 points]

Solution Define the parametric integral

$$F(\varrho) = \iint\limits_{\mathbb{R}^2} \left(\frac{1-\mathrm{e}^{-\varrho xy}}{xy}\right)^2 \mathrm{e}^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\mathbb{R}^2} \frac{1-2\mathrm{e}^{-\varrho xy}+\mathrm{e}^{-2\varrho xy}}{(xy)^2}\mathrm{e}^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y;$$

we have to compute $F(1)$.

In order to get rid of $(xy)^2$ in the denominator, it is natural do differentate F twice. We will compute $F''(\varrho)$, $F'(\varrho)$ and $F(\varrho)$ for $\varrho \in [0,1)$ and then take the limit of F at 1.

For $xy \neq 0$ and $0 \le \varrho < 1$ let

$$f(\varrho,x,y) = \frac{1-2\mathrm{e}^{-\varrho xy}+\mathrm{e}^{-2\varrho xy}}{(xy)^2}\mathrm{e}^{-x^2-y^2}, \quad \text{so} \quad F(\varrho) = \iint\limits_{\mathbb{R}^2} f(\varrho,x,y)\,\mathrm{d}x\mathrm{d}y,$$

$$f_1(\varrho,x,y) = \frac{\partial}{\partial\varrho} f(\varrho,x,y) = \frac{2\mathrm{e}^{-\varrho xy}-2\mathrm{e}^{-2\varrho xy}}{xy}\mathrm{e}^{-x^2-y^2} \quad \text{and} \quad F_1(\varrho) = \iint\limits_{\mathbb{R}^2} f_1(\varrho,x,y)\,\mathrm{d}x\mathrm{d}y$$

and

$$f_2(\varrho,x,y) = \frac{\partial^2}{\partial\varrho^2} f(\varrho,x,y) = (-2\mathrm{e}^{-\varrho xy}+4\mathrm{e}^{-2\varrho xy})\mathrm{e}^{-x^2-y^2} \quad \text{and} \quad F_2(\varrho) = \iint\limits_{\mathbb{R}^2} f_2(\varrho,x,y)\,\mathrm{d}x\mathrm{d}y.$$

Notice that for $0 \le \varrho \le 1-\varepsilon$, f_2 can be dominated as

$$\big|f_2(\varrho,x,y)\big| < 6\mathrm{e}^{2\varrho|xy|-x^2-y^2} = 6\mathrm{e}^{-\varrho(x-y)^2-(1-\varrho)(x^2+y^2)} \le 6\mathrm{e}^{-\varepsilon(x^2+y^2)}$$

where the dominant function $6\mathrm{e}^{-\varepsilon(x^2+y^2)}$ has a finite integral; due to $f(0,x,y) = f_1(0,x,y)$, we can obtain the same dominating function for f_1 and f. This shows that F_2 is continuous, $F_1(\varrho) = \int_0^\varrho f_2(t,x,y)\mathrm{d}t$ and $F(\varrho) = \int_0^\varrho f_1(t,x,y)\mathrm{d}t$ for $\varrho < 1$, so indeed $F_1 = F'$ and $F_2 = F_1'$.

Now we compute

$$F_2(\varrho) = \iint\limits_{\mathbb{R}^2} \frac{\partial^2}{\partial\varrho^2}\left(\frac{1-2\mathrm{e}^{-\varrho xy}+\mathrm{e}^{-2\varrho xy}}{(xy)^2}\mathrm{e}^{-x^2-y^2}\right)\mathrm{d}x\mathrm{d}y = \iint\limits_{\mathbb{R}^2} \big(-2\mathrm{e}^{-\varrho xy}+4\mathrm{e}^{-2\varrho xy}\big)\mathrm{e}^{-x^2-y^2}\big)\,\mathrm{d}x\mathrm{d}y:$$

$$\iint_{\mathbb{R}^2} \mathrm{e}^{-x^2-2txy-y^2}\,\mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2} \mathrm{e}^{-(x+ty)^2-(1-t^2)y^2}\,\mathrm{d}x\mathrm{d}y = \int_{-\infty}^{\infty}\mathrm{e}^{-u^2}\mathrm{d}u \int_{-\infty}^{\infty}\mathrm{e}^{-(1-t^2)y^2}\mathrm{d}y = \sqrt{\pi}\cdot\frac{\sqrt{\pi}}{\sqrt{1-t^2}} = \frac{\pi}{\sqrt{1-t^2}},$$

and similarly

$$\iint_{\mathbb{R}^2} \mathrm{e}^{-x^2-txy-y^2}\,\mathrm{d}x\mathrm{d}y = \frac{\pi}{\sqrt{1-\frac{t^2}{4}}},$$

so

$$F_2(\varrho) = 4\frac{\pi}{\sqrt{1-t^2}} - 2\frac{\pi}{\sqrt{1-\frac{t^2}{4}}}.$$

From $F_1(0) = 0$, we get

$$F_1(\varrho) = \int_0^\varrho F_2(t)\,\mathrm{d}t = 4\pi\arcsin\varrho - 4\pi\arcsin\frac{\varrho}{2}.$$

Then by $F(0) = 0$ and integrating by parts,

$$F(\varrho) = \int_0^\varrho F_1(t)\,\mathrm{d}t = 4\pi \int_0^\varrho \Big(\arcsin(t) - \arcsin(t/2)\Big)\,\mathrm{d}t =$$

$$= 4\pi \left([t\arcsin(t)]_0^\varrho - \int_0^\varrho \frac{t}{\sqrt{1-t^2}}\,\mathrm{d}t - [t\arcsin(t/2)]_0^\varrho + \int_0^\varrho \frac{t}{2\sqrt{1-\frac{1}{4}t^2}}\,\mathrm{d}t \right) =$$

$$= 4\pi \Big(\varrho\arcsin(\varrho) + \sqrt{1-\varrho^2} - \varrho\arcsin(\varrho/2) - \sqrt{4-\varrho^2} + 1\Big)\,.$$

Now we show that F is continuous at $1-0$.
The function $\frac{1-\mathrm{e}^{-u}}{u}$ is bounded for $|u| \le 1$, so in the domain $|xy| \le 1$ we have $f(\varrho,x,y) < C_1\mathrm{e}^{-x^2-y^2}$;
In the domain $|x| \ge 1$, $|y| \ge 1$ we have

$$f(\varrho,x,y) < \left(\frac{1+\mathrm{e}^{\varrho|xy|}}{|xy|}\right)^2 \mathrm{e}^{-x^2-y^2} < \left(\frac{2\mathrm{e}^{|xy|}}{|xy|}\right)^2 \mathrm{e}^{-2|xy|} = \frac{C_2}{|xy|^2}.$$

In the domain $|xy| \ge 1$, $|x| \le 1$ we have

$$f(\varrho,x,y) < \big(1+\mathrm{e}^{\varrho|y|}\big)^2\mathrm{e}^{-y^2} < C_3\mathrm{e}^{-(|y|-2)^2}$$

and similary, for $|xy| \ge 1$, $|y| \le 1$ we have

$$f(\varrho,x,y) < C_3\mathrm{e}^{-(|x|-2)^2}$$

These bounds together provide an integrable dominant function for f, so $F(\varrho)$ is continuous in $[0,1]$.

Finally,

$$F(1) = \lim_{\varrho\to 1-0} F(\varrho) = 4\pi\left(\arcsin 1 - \arcsin\frac{1}{2} - \sqrt{3} + 1\right) = \frac{4\pi^2}{3} - 4(\sqrt{3}-1)\pi.$$

□