The 28th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13th April 2018
Category I

Problem 1 *Every point of the rectangle* $R = [0, 4] \times [0, 40]$ *is coloured using one of four colours. Show that there exist four points in* R *with the same colour that form a rectangle having integer side lengths.* [10 points]

Problem 2 *Find all prime numbers* p *such that* p^3 *divides the determinant*

$$\begin{vmatrix} 2^2 & 1 & 1 & \cdots & 1 \\ 1 & 3^2 & 1 & \cdots & 1 \\ 1 & 1 & 4^2 & & 1 \\ \vdots & \vdots & & \ddots & \\ 1 & 1 & 1 & & (p+7)^2 \end{vmatrix}.$$

[10 points]

Problem 3 *Let* n *be a positive integer and let* $x_1, \dots, x_n$ *be positive real numbers satisfying* $|x_i - x_j| \le 1$ *for all pairs* (i, j) *with* $1 \le i < j \le n$. *Prove that*

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \ge \frac{x_2+1}{x_1+1} + \frac{x_3+1}{x_2+1} + \cdots + \frac{x_n+1}{x_{n-1}+1} + \frac{x_1+1}{x_n+1}.$$

[10 points]

Problem 4 *Determine all possible (finite or infinite) values of*

$$\lim_{x \to -\infty} f(x) - \lim_{x \to +\infty} f(x),$$

if $f\colon \mathbb{R} \to \mathbb{R}$ *is a strictly decreasing continuous function satisfying*

$$f(f(x))^4 - f(f(x)) + f(x) = 1$$

for all $x \in \mathbb{R}$. [10 points]