The 27th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31st March 2017
Category II

Problem 1 *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence with* $a_n \in \{0,1\}$ *for every* n. *Let* $F\colon (-1,1) \to \mathbb{R}$ *be defined by*

$$F(x) = \sum_{n=1}^{\infty} a_n x^n$$

and assume that $F\left(\frac{1}{2}\right)$ *is rational. Show that* F *is the quotient of two polynomials with integer coefficients.*
[10 points]

Problem 2 *Prove or disprove the following statement. If* $g\colon (0,1) \to (0,1)$ *is an increasing function and satisfies* $g(x) > x$ *for all* $x \in (0,1)$, *then there exists a continuous function* $f\colon (0,1) \to \mathbb{R}$ *satisfying* $f(x) < f(g(x))$ *for all* $x \in (0,1)$, *but* f *is not an increasing function.* [10 points]

Problem 3 *Let* $n \geq 2$ *be an integer. Consider the system of equations*

$$x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = \ldots = x_n + \frac{2}{x_1}. \tag{1}$$

1. *Prove that* (1) *has infinitely many real solutions* $(x_1, \ldots, x_n)$ *such that the numbers* $x_1, \ldots, x_n$ *are distinct.*
2. *Prove that every solution* $(x_1, \ldots, x_n)$ *of* (1), *such that the numbers* $x_1, \ldots, x_n$ *are not all equal, satisfies* $|x_1 x_2 \cdots x_n| = 2^{n/2}$.

[10 points]

Problem 4 *A positive integer* t *is called a Jane's integer if* $t = x^3 + y^2$ *for some positive integers* x *and* y. *Prove that for every integer* $n \geq 2$ *there exist infinitely many positive integers* m *such that the set of* n^2 *consecutive integers* $\{m+1, m+2, \ldots, m+n^2\}$ *contains exactly* $n+1$ *Jane's integers.* [10 points]