The 26^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 8^{th} April 2016
Category II

Problem 1 *Let a, b and c be positive real numbers such that $a+b+c=1$. Show that*

$$\left(\frac{1}{a}+\frac{1}{bc}\right)\left(\frac{1}{b}+\frac{1}{ca}\right)\left(\frac{1}{c}+\frac{1}{ab}\right) \geq 1728\,.$$

[10 points]

Solution By using the AM-GM inequality, we deduce that $\frac{1}{a}+\frac{1}{bc}=\frac{1}{a}+\frac{1}{3bc}+\frac{1}{3bc}+\frac{1}{3bc} \geq 4\frac{1}{\sqrt[4]{27ab^3c^3}}$ and $\frac{1}{27}=\left(\frac{a+b+c}{3}\right)^3 \geq abc$. Therefore,

$$\left(\frac{1}{a}+\frac{1}{bc}\right)\left(\frac{1}{b}+\frac{1}{ca}\right)\left(\frac{1}{c}+\frac{1}{ab}\right) \geq 64\cdot\frac{1}{\sqrt[4]{27ab^3c^3}}\frac{1}{\sqrt[4]{27a^3bc^3}}\frac{1}{\sqrt[4]{27a^3b^3c}}=\frac{64}{\sqrt[4]{3^9(abc)^7}}$$

$$\geq \frac{64}{\sqrt[4]{3^9(3^{-3})^7}}=64\sqrt[4]{3^{12}}=64\cdot 27=1728.$$

□

Another Solution If we replace 1 with $a+b+c$ and denote $k=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, then

$$\begin{aligned}&\left(\frac{1}{a}+\frac{a+b+c}{bc}\right)\left(\frac{1}{b}+\frac{a+b+c}{ca}\right)\left(\frac{1}{c}+\frac{a+b+c}{ab}\right)\\ &\quad=\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{a}{bc}\right)\cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{b}{ca}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{c}{ab}\right)\\ &\quad=\left(k+\frac{a}{bc}\right)\left(k+\frac{b}{ca}\right)\left(k+\frac{c}{ab}\right)=k^3+k^2\left(\frac{c}{ab}+\frac{b}{ca}+\frac{a}{bc}\right)+k\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{1}{abc}.\end{aligned}$$

From the inequality between arithmetic and harmonic means for the positive numbers a, b and c it follows that

$$\frac{1}{3}=\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\frac{3}{k} \Leftrightarrow k \geq 9.$$

From the first Solution we already know that $\frac{1}{abc} \geq 27$. In the end we have

$$L.H.S \geq 9^3+9^2\cdot\frac{3}{\sqrt[3]{abc}}+\frac{27}{\sqrt[3]{(abc)^2}}+\frac{1}{abc} \geq 9^3+9^3+27\cdot 9+27=1728.$$

□

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Problem 2 *Let X be a set and let $\mathcal{P}(X)$ be the set of all subsets of X. Let $\mu\colon \mathcal{P}(X) \to \mathcal{P}(X)$ be a map with the property that $\mu(A \cup B) = \mu(A) \cup \mu(B)$ whenever A and B are disjoint subsets of X. Prove that there exists a set $F \subset X$ such that $\mu(F) = F$.* [10 points]

Solution First we will show that μ is monotonic, ie. $\mu(A) \subset \mu(B)$ for any $A \subset B$. Indeed, as $B = (B \setminus A) \cup A$, we have

$$\mu(B) = \mu(B \setminus A) \cup \mu(A) \supset \mu(A)\,.$$

Now let $\mathcal{F} = \big\{A \subset X : \mu(A) \subset A\big\}$ and $F = \bigcap \mathcal{F} = \big\{x \in X : \ \forall_{A \in \mathcal{F}}\ x \in A\big\}$. We have of course $F = \bigcap \mathcal{F} \subset A$ for any $A \in \mathcal{F}$, hence by monotonicity $\mu(F) \subset \mu(A)$. But $\mu(A) \subset A$, so $\mu(F) \subset A$ for all $A \in \mathcal{F}$, which gives $\mu(F) \subset \bigcap \mathcal{F} = F$.

On the other hand the monotonicity of μ gives $\mu\big(\mu(F)\big) \subset \mu\big(F\big)$, hence $\mu(F) \in \mathcal{F}$, so $F = \bigcap \mathcal{F} \subset \mu(F)$.

Both inclusions give the equality $\mu(F) = F$. $\square$

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Problem 3 *For $n \geq 3$ find the eigenvalues (with their multiplicities) of the $n \times n$ matrix*

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & \dots & \dots & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & \dots & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & \dots & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 1 \end{bmatrix}.$$

[10 points]

Solution Notice that $A = B^2$ with

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

Lemma *If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.*
Proof $A^2 v = A(Av) = A\lambda v = \lambda A v = \lambda^2 v$. □

It is sufficient to determine eigenvalues of B. Characteristic polynomial $S_n(\lambda) = det(\lambda I - B)$ of matrix B satisfies the following reccurence relation
$S_1 = \lambda, S_2 = \lambda^2 - 1$ and

$$S_n(\lambda) = \lambda S_{n-1}(\lambda) - S_{n-2}(\lambda), \quad n \geq 3.$$

We have

$$S_n(\lambda) = U_n\left(\frac{\lambda}{2}\right),$$

with U_n being a Chebyshev polynomial of the second kind which is given by the reccurence relation

$$U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0, \quad U_0(x) = 1, U_1(x) = 2x,$$

or explicitly with

$$U_n(x) = \frac{\sin((n+1)\arccos x)}{\sin(\arccos x)}, \quad |x| < 1.$$

Lemma (Gershgorin circle theorem) *Every eigenvalue of a complex $n \times n$ matrix A lies within at least one of the disks*

$$D = \{z \in \mathbb{C} : |z - a_{ii}| \leq R_i\},$$

with $R_i = \displaystyle\sum_{j=1, j\neq i}^{n} |a_{ij}|$.

Proof Let λ be an eigenvalue of A and x its corresponding eigenvector. Choose i such that $|x_i| = \max_j |x_j|$. Since $x \neq 0$, $|x_i| > 0$. From $Ax = \lambda x$, looking at the ith component we have

$$(\lambda - a_{ii})x_i = \sum_{j\neq i} a_{ij}x_j.$$

Taking the norm of both sides gives

$$|\lambda - a_{ii}| = \left| \sum_{j \neq i} \frac{a_{ij} x_j}{x_i} \right| \leq \sum_{j \neq i} |a_{ij}|.$$

□

From Gershgorin circle theorem we conclude that each eigenvalue λ_i of B satisfies $|\lambda_i| \leq 1$. For $\frac{\lambda}{2} = \cos\theta$ we get

$$U_n\left(\frac{\lambda}{2}\right) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

From equation $S_n(\lambda) = 0$ it follows that $\sin((n+1)\arccos\frac{\lambda}{2}) = 0$,

$$(n+1)\arccos\frac{\lambda}{2} = k\pi, \quad k \in \mathbb{Z}. \tag{1}$$

We need first n solutions of the equation (1). Therefore,

$$\lambda_k(B) = 2\cos\frac{k\pi}{n+1}, \quad k = 1, \ldots, n$$

and

$$\lambda_k(A) = 4\cos^2\frac{k\pi}{n+1}, \quad k = 1, \ldots, n.$$

□

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Problem 4 *Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuously differentiable function satisfying*

$$f(x) = \int_{x-1}^{x} f(t)\,\mathrm{d}t$$

for all $x \geq 1$. Show that f has bounded variation on $[1, \infty)$, i.e.

$$\int_1^\infty |f'(x)|\,\mathrm{d}x < \infty\,.$$

[10 points]

Solution Since f is continuous, the right-hand side of

$$f(x) = \int_{x-1}^{x} f(t)dt$$

is differentiable and the derivative is equal to $f(x) - f(x-1)$. So, f is differentiable on $(1, +\infty)$ and $f'(x) = f(x) - f(x-1)$. Let us denote $A_n := \max\{f(x) : \; x \in [n, n+1]\}$, $B_n := \min\{f(x) : \; x \in [n, n+1]\}$ and $d_n = A_n - B_n$ for $n = 0, 1, \ldots$. Then $|f'| \leq d_n$ on $[n+1, n+2]$ and

$$\int_1^{+\infty} |f'(x)|dx = \sum_{n=1}^{\infty} \int_n^{n+1} |f'(x)|dx \leq \sum_{n=1}^{\infty} d_{n-1}.$$

So, it is sufficient to show that $\sum_{n=1}^{\infty} d_{n-1}$ converges. We will complete the proof in three steps.

Claim 1. *$A_n \leq A_{n-1}$ and $B_n \geq B_{n-1}$ for all n and if there is an equality for some n, then $f \equiv A_n$ on $[n, +\infty)$.*

Claim 1 implies that d_n is a nonincreasing sequence of nonnegative numbers. If $d_{n_0} = 0$ for some $n_0 \in \mathbb{N}$, then $\sum d_{n-1}$ obviously converges. Otherwise, $d_n > 0$ for all n and we complete the proof by showing the following two Claims.

Claim 2. *It holds that $A_{n+2} \leq A_n - \frac{d_n^5}{128 d_{n-1}^4}$, $B_{n+2} \geq B_n + \frac{d_n^5}{128 d_{n-1}^4}$, and $d_{n+2} \leq d_n - \frac{d_n^5}{64 d_{n-1}^4}$.*

Claim 3. *$\sum_{n=1}^{\infty} d_{n-1} < +\infty$.*

Proof of Claim 1. Let us assume $A_n > A_{n-1}$. There exists $x \in [n, n+1]$ such that $f(x) = A_n$ and since f is continuous, the set $\{x \in [n, n+1] : f(x) = A_n\}$ has a minimum m. But then

$$f(m) = \int_{m-1}^{m} f(x)dx < \int_{m-1}^{m} A_n dx = A_n,$$

contradiction. If $A_n = A_{n-1}$ and let $m = \min\{x \in [n, n+1] : f(x) = A_n\}$, then we have

$$A_n = f(m) = \int_{m-1}^{m} f(t)dt \leq \int_{m-1}^{m} A_n dt = A_n$$

and it follows that $f(x) = A_n$ for all $x \in [m-1, m]$. Hence, $m = n$ and $f \equiv A_n$ on $[n-1, n]$ and by induction on $[n-1, +\infty)$. The inequalities for B's can be proven analogously. □

Proof of Claim 2. Let us fix $n \geq 1$ and show the first inequality. Since $|f'| \leq d_{n-1}$ on $[n, n+1]$ and f attains values A_n and B_n on $[n, n+1]$, it follows that $\int_n^{n+1} f(t)dt \leq A_n - \frac{d_n^2}{2d_{n-1}} =: K_1$ (graph of f must connect lines $y \equiv A_n$, $y \equiv B_n$ and this connection must lie below the straight line with tangent d_{n-1}). It follows that $f(n+1) \leq K_1$ and since $f \leq A_n$ and $f' \leq d_n$ on $[n+1, n+2]$, we have $f \leq h_1$ on $[n+1, n+2]$, where

$$h_1(x) := \begin{cases} K_1 + d_n(x - n - 1) & \text{for } x \in [n+1, x_1] \\ A_n & \text{for } x \in [x_1, n+2] \end{cases}, \quad x_1 := n + 1 + \frac{d_n}{2d_{n-1}}.$$

Further,

$$f(n+2) \leq \int_{n+1}^{n+2} h_1(x) = A_n - \frac{d_n^3}{8d_{n-1}^2} =: K_2$$

and since $f \leq A_n$ and $f' \leq d_{n+1} \leq d_n$ on $[n+2, n+3]$, we obtain that $f \leq h_2$ on $[n+2, n+3]$, where

$$h_2(x) := \begin{cases} K_2 + d_n(x - n - 2) & \text{for } x \in [n+2, x_2] \\ A_n & \text{for } x \in [x_2, n+3] \end{cases}, \quad x_2 := n + 2 + \frac{d_n^2}{8d_{n-1}^2}$$

(x_1 and x_2 are taken in such a way that h_1 and h_2 are continuous on $[n+1, n+2]$, resp. $[n+2, n+3]$). Clearly, $K_2 \geq K_1$, therefore $f(x) \leq h_1(x) \leq h_2(x+1)$ on $[n+1, n+2]$. It follows that for all $x \in [n+2, n+3]$ we have

$$f(x) = \int_{x-1}^{x} f(t) \leq \int_{x-1}^{n+2} h_2(t+1) + \int_{n+2}^{x} h_2(t) = \int_{n+2}^{n+3} h_2(x) = A_n - \frac{d_n^5}{128 d_{n-1}^4}.$$

Hence, $A_{n+2} \leq A_n - \frac{d_n^5}{128 d_{n-1}^4}$. The inequality for B_{n+2} is similar and inequality for d_{n+2} is then an immediate consequence. □

Proof of Claim 3. Dividing the inequality for d_{n+2} by d_n we obtain

$$\frac{d_{n+2}}{d_n} \leq 1 - \frac{d_n^4}{64 d_{n-1}^4} \leq 1 - \frac{d_n^4}{64 d_{n-2}^4}$$

and therefore

$$\frac{d_{n+2}}{d_{n-2}} = \frac{d_{n+2}}{d_n} \cdot \frac{d_n}{d_{n-2}} \leq \left(1 - \frac{d_n^4}{64 d_{n-2}^4}\right) \frac{d_n}{d_{n-2}} \tag{1}$$

Differentiating $g(x) = x(1 - x^4/64)$ we obtain $g'(x) = 1 - 5x^4/64 > 0$ on $[0, 1]$, hence the right-hand side of (1) is maximal if $\frac{d_n}{d_{n-2}} = 1$, i.e.

$$\frac{d_{n+2}}{d_{n-2}} \leq 1 - \frac{1}{64} = \frac{63}{64}.$$

It follows that each of the sequences $(d_{4k+i})_{k=0}^{\infty}$, $i = 0, 1, 2, 3$ is dominated by $d_i \frac{63^n}{64^n}$, hence $\sum d_i < +\infty$. □