The 26th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 8th April 2016
Category II

Problem 1 *Let a, b and c be positive real numbers such that $a+b+c=1$. Show that*

$$\left(\frac{1}{a}+\frac{1}{bc}\right)\left(\frac{1}{b}+\frac{1}{ca}\right)\left(\frac{1}{c}+\frac{1}{ab}\right) \geq 1728\,.$$

[10 points]

Problem 2 *Let X be a set and let $\mathcal{P}(X)$ be the set of all subsets of X. Let $\mu\colon \mathcal{P}(X) \to \mathcal{P}(X)$ be a map with the property that $\mu(A \cup B) = \mu(A) \cup \mu(B)$ whenever A and B are disjoint subsets of X. Prove that there exists a set $F \subset X$ such that $\mu(F) = F$.* [10 points]

Problem 3 *For $n \geq 3$ find the eigenvalues (with their multiplicities) of the $n \times n$ matrix*

$$\begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 0 & \dots & \dots & 0 & 0\\
0 & 2 & 0 & 1 & 0 & 0 & \dots & \dots & 0 & 0\\
1 & 0 & 2 & 0 & 1 & 0 & \dots & \dots & 0 & 0\\
0 & 1 & 0 & 2 & 0 & 1 & \dots & \dots & 0 & 0\\
0 & 0 & 1 & 0 & 2 & 0 & \dots & \dots & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 2 & \dots & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 2 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & 0 & 1
\end{bmatrix}.$$

[10 points]

Problem 4 *Let $f\colon [0,\infty) \to \mathbb{R}$ be a continuously differentiable function satisfying*

$$f(x) = \int_{x-1}^{x} f(t)\,\mathrm{d}t$$

for all $x \geq 1$. Show that f has bounded variation on $[1,\infty)$, i.e.

$$\int_1^\infty |f'(x)|\,\mathrm{d}x < \infty\,.$$

[10 points]