The 26th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 8th April 2016
Category I

Problem 1 *Let $f\colon \mathbb{R} \to (0,\infty)$ be a continuously differentiable function. Prove that there exists $\xi \in (0,1)$ such that*

$$\mathrm{e}^{f'(\xi)} f(0)^{f(\xi)} = f(1)^{f(\xi)}\,.$$

[10 points]

Problem 2 *Find all positive integers n such that $\varphi(n)$ divides n^2+3.*
($\varphi(n)$ denotes Euler's totient function, i.e. the number of positive integers $k \le n$ coprime to n.) [10 points]

Problem 3 *Let $d \ge 3$ and let $A_1 \dots A_{d+1}$ be a simplex in $\mathbb{R}^d$. (A simplex is the convex hull of $d+1$ points not lying in a common hyperplane.) For every $i = 1, \dots, d+1$ let O_i be the circumcentre of the face $A_1 \dots A_{i-1} A_{i+1} \dots A_{d+1}$, i.e. O_i lies in the hyperplane $A_1 \dots A_{i-1} A_{i+1} \dots A_{d+1}$ and it has the same distance from all points $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_{d+1}$. For each i draw a line through A_i perpendicular to the hyperplane $O_1 \dots O_{i-1} O_{i+1} \dots O_{d+1}$. Prove that either these lines are parallel or they have a common point.* [10 points]

Problem 4 *Find the value of the sum $\sum\limits_{n=1}^{\infty} A_n$, where*

$$A_n = \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \frac{1}{k_1^2}\,\frac{1}{k_1^2+k_2^2} \cdots \frac{1}{k_1^2+\cdots+k_n^2}\,.$$

[10 points]