The 25th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 27th March 2015
Category II

Problem 1 *Let A and B be two 3×3 matrices with real entries. Prove that*

$$A - \left(A^{-1} + (B^{-1} - A)^{-1}\right)^{-1} = ABA,$$

provided all the inverses appearing on the left-hand side of the equality exist. [10 points]

Solution Let A, B be elements of an arbitrary associative algebra with unit. We have:

$$\begin{aligned}
(A^{-1} + (B^{-1} - A)^{-1})^{-1} &= \Big(A^{-1}(B^{-1} - A)(B^{-1} - A)^{-1} + A^{-1}A(B^{-1} - A)^{-1}\Big)^{-1} \\
&= \Big(A^{-1}\Big((B^{-1} - A) + A\Big)(B^{-1} - A)^{-1}\Big)^{-1} \\
&= \Big(A^{-1}B^{-1}(B^{-1} - A)^{-1}\Big)^{-1} \\
&= (B^{-1} - A)BA \\
&= A - ABA,
\end{aligned}$$

provided all the inverses appearing here exist, from which the desired equality follows. □

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Problem 2 *Determine all pairs* (n, m) *of positive integers satisfying the equation*

$$5^n = 6m^2 + 1\,.$$

[10 points]

Solution Looking $\pmod 3$ we get that n is even, say $n = 2k$. We arrive at Pell's equation of the form $x^2 - 6m^2 = 1$, where $x = 5^k$. All positive solutions to this equation are given by the formula

$$x_\nu = \frac{(5+2\sqrt{6})^\nu + (5-2\sqrt{6})^\nu}{2}, \quad \nu = 1, 2, \ldots$$

We get

$$2 \cdot 5^k = (5+2\sqrt{6})^\nu + (5-2\sqrt{6})^\nu,$$

if $k = 1$ then we have a solution $(k, \nu) = (1, 1)$ which corresponds to $(n, m) = (2, 2)$. Suppose that $k \geq 2$ looking $\pmod{25}$ we get $(5+2\sqrt{6})^\nu + (5-2\sqrt{6})^\nu \equiv 10\nu(2\sqrt{6})^{\nu-1} \pmod{25}$ when ν is odd, and $(5+2\sqrt{6})^\nu + (5-2\sqrt{6})^\nu \equiv 2(2\sqrt{6})^\nu \pmod{25}$ when ν is even, thus RHS is divisible by 25 iff $\nu \equiv 5 \pmod{10}$. But in that case we have

$$2 \cdot 5^2 \cdot 1901 = (5+2\sqrt{6})^5 + (5-2\sqrt{6})^5 | (5+2\sqrt{6})^\nu + (5-2\sqrt{6})^\nu,$$

and therefore $(5+2\sqrt{6})^\nu + (5-2\sqrt{6})^\nu$ cannot be of the form $2 \cdot 5^k$.

The only solution to our equation is $(n, m) = (2, 2)$.

□

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Problem 3 *Determine the set of real values of x for which the following series converges, and find its sum:*

$$\sum_{n=1}^{\infty}\left(\sum_{\substack{k_1,\dots,k_n\geq 0\\ 1\cdot k_1+2\cdot k_2+\cdots+n\cdot k_n=n}}\frac{(k_1+\cdots+k_n)!}{k_1!\cdot\ldots\cdot k_n!}\,x^{k_1+\cdots+k_n}\right).$$

[10 points]

Solution

Lemma *(Faà di Bruno's formula)*

$$\big(f(g(t))\big)^{(n)}=\sum_{\substack{k_1,\dots,k_n\geq 0\\ 1\cdot k_1+2\cdot k_2+\ldots+n\cdot k_n=n}}\frac{n!}{k_1!\ldots k_n!\,1!^{k_1}\ldots n!^{k_n}}f^{(k_1+\ldots+k_n)}(g(t))\cdot(g'(t))^{k_1}\cdot\ldots\cdot(g^{(n)}(t))^{k_n}.$$

Using this formula, after differentiating the following identity n times

$$\frac{1}{\frac{1}{t}-1}=\frac{t}{1-t}=-1-\frac{1}{t-1}$$

one has

$$\sum\frac{n!}{k_1!\ldots k_n!\,1!^{k_1}\ldots n!^{k_n}}\frac{(-1)^{k_1+\ldots+k_n}(k_1+\ldots+k_n)!}{\left(\frac{1}{t}-1\right)^{k_1+\ldots+k_n+1}}\prod_{j=1}^{n}\left(\frac{(-1)^j j!}{t^{j+1}}\right)^{k_j}=\frac{(-1)^{n+1}n!}{(t-1)^{n+1}}.$$

After simplifications we have

$$\sum\frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!}\frac{1}{(t-1)^{k_1+\ldots+k_n}}=\frac{t^{n-1}}{(t-1)^n}.$$

Put $t=\frac{1}{x}+1$. Then

$$\sum\frac{(k_1+\ldots+k_n)!}{k_1!\ldots k_n!}x^{k_1+\ldots+k_n}=x(x+1)^{n-1}.$$

So the series $\sum_{n=1}^{\infty}x(x+1)^{n-1}$ converges iff $-2<x\leq 0$, and its sum equals 0 for $x=0$ and equals $\frac{x}{1-(x+1)}=-1$ for $-2<x<0$. □

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Problem 4 *Find all continuously differentiable functions* $f\colon \mathbb{R} \to \mathbb{R}$, *such that for every* $a \geq 0$ *the following relation holds:*

$$\iiint\limits_{D(a)} x f\left(\frac{ay}{\sqrt{x^2+y^2}}\right) \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{\pi a^3}{8}\left(f(a) + \sin a - 1\right), \tag{1}$$

where $D(a) = \left\{(x,y,z) : x^2+y^2+z^2 \leq a^2, |y| \leq \frac{x}{\sqrt{3}}\right\}$. [10 points]

Solution Clearly, for $a = 0$ the condition is always satisfied. Let us mark the integral as J. Transformation to the spherical coordinates r, θ, φ gives us

$$J = \int\limits_{-\frac{\pi}{6}}^{\frac{\pi}{6}} f(a\sin\varphi)\cos\varphi\, d\varphi \int\limits_0^{\pi} \sin^2\theta\, d\theta \int\limits_0^a r^3 dr.$$

Taking $t = a\sin\varphi$, we obtain $J = \frac{\pi a^3}{8} \int\limits_{-\frac{a}{2}}^{\frac{a}{2}} f(t)dt$. Then, equality (1) takes the form

$$\int\limits_{-\frac{a}{2}}^{\frac{a}{2}} f(t)dt = f(a) + \sin a - 1. \tag{2}$$

By differentiation of the both parts of (2) with respect to a, one comes to the following differential equation

$$\frac{1}{2}(f(a/2) + f(-a/2)) = f'(a) + \cos a, \qquad a > 0 \tag{3}$$

and this condition is equivalent to (1), since f is continuously differentiable.

Take an arbitrary $f \in C^1([0,+\infty))$ and define f on negative semiaxis by

$$f(-a/2) = 2f'(a) + 2\cos a - f(a/2), \qquad a > 0. \tag{4}$$

Clearly, any such f satisfies (3). It remains to investigate, under what conditions it is continuously differentiable on $\mathbb{R}$. From continuity in 0 and (2) we have $f(0) = 1$. From (3) we have $f'_+(0) = 0$ (taking $a \to 0+$). Moreover, $f \in C^1((-\infty,0))$ if and only if $f \in C^2((0,+\infty))$ by (3). Finally, $f'(0)$ exists if and only if $f''_+(0) = 0$.

Let $f \in C^2([0,+\infty))$ satisfies $f(0) = 1$, $f'_+(0) = f''_+(0) = 0$, then its extension to $\mathbb{R}$ defined by (4) satisfies (1). These are all solutions to (1).

□