The 25^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 27^{th} March 2015
Category I

Problem 1 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$. Prove that there exists $x \in [0,1]$ such that*

$$\frac{4}{\pi}\big(f(1) - f(0)\big) = (1 + x^2) f'(x).$$

[10 points]

Solution Consider differentiable function $\psi(x) = \phi(\tan x)$ on $[0, \frac{\pi}{4}]$. In accordance with the Lagrange theorem, there exists $\xi \in [0, \frac{\pi}{4}]$ such that

$$\psi(\frac{\pi}{4}) - \psi(0) = \frac{\pi}{4}\psi'(\xi).$$

Correspondingly,

$$\phi(1) - \phi(0) = \frac{\pi}{4}(1 + (\tan \xi)^2)\phi'(\tan \xi).$$

Taking $z = \tan \xi$, one obtains the result.

$\square$

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Problem 2 *Consider the infinite chessboard whose rows and columns are indexed by positive integers. Is it possible to put a single positive rational number into each cell of the chessboard so that each positive rational number appears exactly once and the sum of every row and of every column is finite?* [10 points]

Solution Yes, it is possible. Let $A = \{1/, 1/2, 1/4, 1/8, \dots\}, B = \mathbb{Q}_+ \setminus A$. Write elements of B on diagonal, and elements of A in other squares. $\square$

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Problem 3 *Let* $P(x) = x^{2015} - 2x^{2014} + 1$ *and* $Q(x) = x^{2015} - 2x^{2014} - 1$. *Determine for each of the polynomials* P *and* Q *whether it is a divisor of some nonzero polynomial* $c_0 + c_1 x + \cdots + c_n x^n$ *whose coefficients* c_i *are all in the set* $\{1, -1\}$. [10 points]

Solution Answer. The polynomial $P(x)$ is a factor of such a polynomial and the polynomial $Q(x)$ isn't.

Let $n \geq 2$ be an integer. Multiplying $P_n(x) = x^n - 2x^{n-1} + 1$ by $F_n(x) = 1 + x + \cdots + x^{n-1}$, we obtain the polynomial

$$P_n(x)F_n(x) = F_n(x) - 2x^{n-1}F_n(x) + x^n F_n(x) = \sum_{i=0}^{n-2} x^i - \sum_{i=n-1}^{2n-2} x^i + x^{2n-1},$$

which has the required form.

Assume that $Q_n(x) = x^n - 2x^{n-1} - 1$ divides some $G(x) = \sum_{j=0}^{d} g_j x^j$, where $d > n$ and $g_j \in \{-1, 1\}$ for $j = 0, \ldots, d$. Since $Q_n(2) = -1 < 0$, the polynomial Q_n has a real root $\alpha > 2$. Then $G(\alpha) = \sum_{j=0}^{d} g_j \alpha^j = 0$. This yields

$$\alpha^d = \left| -\sum_{j=0}^{d-1} \frac{g_j}{g_d} \alpha^j \right| \leq \sum_{j=0}^{d-1} \alpha^j = \frac{\alpha^d - 1}{\alpha - 1} < \alpha^d - 1,$$

which is a contradiction. $\square$

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Problem 4 *Let m be a positive integer and let p be a prime divisor of m. Suppose that the complex polynomial $a_0 + a_1x + \ldots + a_nx^n$ with $n < \frac{p}{p-1}\varphi(m)$ and $a_n \neq 0$ is divisible by the cyclotomic polynomial $\Phi_m(x)$. Prove that there are at least p nonzero coefficients a_i.*

The cyclotomic polynomial $\Phi_m(x)$ is the monic polynomial whose roots are the m-th primitive complex roots of unity. Euler's totient function $\varphi(m)$ denotes the number of positive integers less than or equal to m which are coprime to m. [10 points]

Solution In view of the identity $\Phi_{p^2k}(x) = \Phi_{pk}(x^p)$ we can assume that p^2 does not divide m. Assume $m = pk$ where p does not divide k.

Take an arbitrary residue class r modulo p; we will show that there is an index ℓ such that $\ell \equiv r \pmod p$ and $a_\ell \neq 0$. Then the statement will follow.

Let $\varepsilon = e^{2\pi i/m}$ and $\varrho = e^{2\pi i/p} = \varepsilon^k$.

Since $\deg f(x) < \frac{p}{p-1}\varphi(m) = p \cdot \varphi(k) = \deg \Phi_k(x^p)$, there is a root w of $\Phi_k(x^p)$ which is not a root of $f(x)$. The roots of $\Phi_k(x^p)$ have multiplicative orders k and $pk = m$. On the other hand, all unit complex numbers with order m are roots of $\Phi_m(x)$ and thus they are roots of $f(x)$. So, the order of w must be k; we can see that $w = \varepsilon^s$ with some integer s satisfying $\gcd(s, m) = p$.

Now consider the sum

$$S = \sum_{j=0}^{p-1} \varepsilon^{-jrk} f(\varepsilon^{jk+s}).$$

The exponents $k+s, 2k+s, \ldots, (p-1)k+s$ are co-prime to k. Moreover, since p divides s and p does not divide k, the numbers $k+s, 2k+s, \ldots, (p-1)k+s$ are not divide by p. Hence, $k+s, 2k+s, \ldots, (p-1)k+s$ are all co-prime to $pk = m$. Then $\varepsilon^{k+s}, \varepsilon^{2k+s}, \ldots, \varepsilon^{(p-1)k+s}$ are all primitive mth roots of unity, so they are roots of $f(x)$. Therefore,

$$S = \sum_{j=0}^{p-1} \varepsilon^{-jrk} f(\varepsilon^{jk+s}) = f(\varepsilon^s) + \sum_{j=1}^{p-1} \varepsilon^{-jrk} \underbrace{f(\varepsilon^{jk+s})}_{=0} = f(w) \neq 0. \tag{1}$$

On the other hand, exchanging the sums,

$$S = \sum_{j=0}^{p-1} \varepsilon^{-jrk} f(\varepsilon^{jk+s}) = \sum_{j=0}^{p-1} \varepsilon^{-jrk} \sum_{\ell=0}^{n} a_\ell \varepsilon^{(jk+s)\ell} = \sum_{\ell=0}^{n} a_\ell \varepsilon^{s\ell} \sum_{j=0}^{p-1} \varepsilon^{jk(\ell-r)} = \sum_{\ell=0}^{n} a_\ell \varepsilon^{s\ell} \sum_{j=0}^{p-1} \varrho^{j(\ell-r)}.$$

It is well-known that for every integer t,

$$\sum_{j=0}^{p-1} \varrho^{jt} = \begin{cases} p & \text{if } p|t; \\ 0 & \text{otherwise.} \end{cases}$$

So,

$$S = \sum_{\ell=0}^{n} a_\ell \varepsilon^{r\ell} \sum_{j=0}^{p-1} \varrho^{j(\ell-r)} = \sum_{\ell \equiv r \pmod p} a_\ell \varepsilon^{r\ell} p. \tag{2}$$

Comparing (1) and (2) we can see that there must be an index $\ell \equiv r \pmod p$ such that $a_\ell \neq 0$. $\square$