The 25th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 27th March 2015
Category II

Problem 1 *Let A and B be two 3×3 matrices with real entries. Prove that*

$$A - \left(A^{-1} + (B^{-1} - A)^{-1}\right)^{-1} = ABA,$$

provided all the inverses appearing on the left-hand side of the equality exist. [10 points]

Problem 2 *Determine all pairs (n, m) of positive integers satisfying the equation*

$$5^n = 6m^2 + 1\,.$$

[10 points]

Problem 3 *Determine the set of real values of x for which the following series converges, and find its sum:*

$$\sum_{n=1}^{\infty} \left(\sum_{\substack{k_1,\dots,k_n \ge 0 \\ 1\cdot k_1 + 2\cdot k_2 + \cdots + n\cdot k_n = n}} \frac{(k_1 + \cdots + k_n)!}{k_1! \cdot \ldots \cdot k_n!}\, x^{k_1 + \cdots + k_n} \right).$$

[10 points]

Problem 4 *Find all continuously differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$, such that for every $a \ge 0$ the following relation holds:*

$$\iiint\limits_{D(a)} x f\left(\frac{ay}{\sqrt{x^2 + y^2}}\right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \frac{\pi a^3}{8}\left(f(a) + \sin a - 1\right),$$

where $D(a) = \left\{(x, y, z) : x^2 + y^2 + z^2 \le a^2, |y| \le \dfrac{x}{\sqrt{3}}\right\}$. [10 points]