The 23rd Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 12th April 2013
Category I

Problem 1 *Let $f\colon [0,\infty) \to \mathbb{R}$ be a differentiable function with $|f(x)| \leq M$ and $f(x)f'(x) \geq \cos x$ for $x \in [0,\infty)$, where $M > 0$. Prove that $f(x)$ does not have a limit as $x \to \infty$.*

Solution Consider a function $F\colon [0,\infty) \to \mathbb{R}$ given by

$$F(x) := f^2(x) - 2\sin x.$$

Then:

- $|F(x)| \leqslant f^2(x) + 2|\sin x| \leqslant M + 2.$
- $F'(x) = 2f(x)f'(x) - 2\cos x \geqslant 0.$

Hence we infer that F is increasing and bounded. Let

$$x_n = \begin{cases} n\pi & \text{if} \quad n = 2k-1, \\ n\pi + \frac{\pi}{2} & \text{if} \quad n = 2k. \end{cases}$$

Then $(F(x_n))$ is increasing and bounded and hence convergent. Assume on the contrary that $\lim\limits_{x\to\infty} f(x)$ exists. In turn, this implies that $\lim\limits_{n\to\infty} f^2(x_n)$ exists. Thus the sequence $F(x_n) - f^2(x_n)$ is convergent. But

$$F(x_n) - f^2(x_n) = -2\sin(x_n)\,.$$

Consequently we get that the sequence $(\sin(x_n))$ is convergent. This contradicts the fact that $(\sin(x_n))$ is not convergent since

$$\sin(x_n) = \begin{cases} 0 & \text{if} \quad n = 2k-1, \\ 1 & \text{if} \quad n = 2k. \end{cases}$$

□

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Problem 2 *Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real 10×10 matrices such that $a_{ij} = b_{ij} + 1$ for all i, j and $A^3 = 0$. Prove that $\det B = 0$.*

Solution Let H be the matrix 10×10 consisting of units. Then $A = B + H$. As $A^3 = 0$ then

$$B^3 = (A - H)^3 = A^3 + \text{a sum of 7 matrices of the rank} \leq 1.$$

Therefore rank $B^3 \leq 7$. Since B is of size 10×10, B is degenerate. □

Problem 3 *Let S be a finite set of integers. Prove that there exists a number c depending on S such that for each non-constant polynomial f with integer coefficients the number of integers k satisfying $f(k) \in S$ does not exceed* $\max(\deg f, c)$.

Solution For each set $T \subseteq \mathbb{Z}$ let $N(f,T)$ denote the number of distinct integers k for which $f(k) \in T$. Suppose that the cardinality of S is at least 2 and suppose for some two elements $s_1 \neq s_2$ of S the equations $f(x) = s_1$ and $f(x) = s_2$ both have integer solutions, say, $x = k_1$ and $x = k_2$, respectively. (Otherwise, we immediately obtain $N(f,S) \leq \deg f$.) Put $d = d(S)$ for the difference between the largest and the smallest elements of S. We claim that then $N(f,S) \leq 4d(S)$.

Indeed, if for some $k \in \mathbb{Z}$ we have $f(k) = s \in S$, where $s \neq s_1$ (and so $k \neq k_1$), then $k - k_1$ divides the integer $f(k) - f(k_1) = s - s_1$. Thus $|k - k_1| \leq |s - s_1| \leq d$. Clearly, there are at most $2d$ of such integers k (since $k \neq k_1$), so $N(f, S \setminus \{s_1\}) \leq 2d$. By the same argument, we must have $N(f, S \setminus \{s_2\}) \leq 2d$. Since S is contained in the union of the sets $S \setminus \{s_1\}$ and $S \setminus \{s_2\}$, we deduce that

$$N(f,S) \leq N(f, S \setminus \{s_1\}) + N(f, S \setminus \{s_2\}) \leq 2d + 2d = 4d.$$

Therefore, $N(f,S) \leq \max(\deg f, 4d(S))$. $\square$

Problem 4 *Let n and k be positive integers. Evaluate the following sum*

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k}$$

where $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$.

Solution We show that

$$\sum_{j=0}^{k} \binom{k}{j}^2 \binom{n+2k-j}{2k} = \binom{n+k}{k}^2. \tag{1}$$

Multiplying equation (1) by $\dfrac{(2k)!n!}{(n+k)!k!}$ we get

$$\sum_{j=0}^{k} \binom{k}{j} \frac{k!}{j!(k-j)!} \frac{(n+2k-j)!}{(2k)!(n-j)!} \frac{(2k)!n!}{(n+k)!k!} = \sum_{j=0}^{k} \binom{k}{j} \frac{n!}{j!(n-j)!} \frac{(n+2k-j)!}{(n+k)!(k-j)!}$$
$$= \sum_{j=0}^{k} \binom{k}{j} \binom{n}{j} \binom{n+2k-j}{k-j}. \tag{2}$$

On the right side in the formula (1) after multiplying we obtain

$$\binom{n+k}{k} \frac{(n+k)!}{k!n!} \frac{(2k)!n!}{(n+k)!k!} = \binom{n+k}{k}\binom{2k}{k}.$$

Applying Cauchy identity

$$\binom{m+n}{k} = \sum_{r=0}^{k} \binom{n}{r}\binom{m}{k-r},$$

to formula (2) we have

$$\sum_{j=0}^{k} \binom{k}{j}\binom{n}{j} \sum_{r=0}^{k-j} \binom{n-j}{r}\binom{2k}{k-j-r}. \tag{3}$$

By changing the order of summation in formula (3) putting $s = r + j$ we get

$$\sum_{j=0}^{k} \binom{k}{j}\binom{n}{j} \sum_{s=j}^{k} \binom{n-j}{s-j}\binom{2k}{k-s} =$$
$$\sum_{j=0}^{k} \binom{k}{j}\binom{n}{j} \sum_{s=0}^{k} \binom{n-j}{s-j}\binom{2k}{k-s}. \tag{4}$$

Once again by changing the order of summation in formula (4) it follows

$$\sum_{s=0}^{k} \binom{2k}{k-s} \sum_{j=0}^{s} \binom{k}{j}\binom{n}{j}\binom{n-j}{s-j}.$$

On account of the Cauchy identity we have

$$\binom{2k}{k} \sum_{s=0}^{k} \binom{n}{s}\binom{k}{k-s}.$$

Finally we show that

$$\binom{2k}{k-s}\sum_{j=0}^{s}\binom{k}{j}\binom{n}{j}\binom{n-j}{s-j}=\binom{2k}{k}\binom{n}{s}\binom{k}{k-s}.$$

By applying well-known formula

$$\binom{n}{m}\binom{m}{k}=\binom{n}{k}\binom{n-k}{m-k}.$$

it follows

$$\begin{aligned}\binom{2k}{k-s}\sum_{j=0}^{s}\binom{k}{j}\binom{n}{j}\binom{n-j}{s-j}&=\binom{2k}{k+s}\sum_{j=0}^{s}\binom{k}{j}\binom{n}{s}\binom{s}{j}=\binom{2k}{k+s}\binom{n}{s}\sum_{j=0}^{s}\binom{k}{j}\binom{s}{s-j}\\&=\binom{2k}{k+s}\binom{n}{s}\binom{k+s}{s}=\binom{2k}{k+s}\binom{n}{s}\binom{k+s}{k}=\binom{n}{s}\binom{2k}{k}\binom{2k-k}{k+s-k}\\&=\binom{n}{s}\binom{2k}{k}\binom{k}{s}=\binom{n}{s}\binom{2k}{k}\binom{k}{k-s}.\end{aligned}$$

This completes the proof of Li-en-Szua formula. □