The 21st Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31st March 2011
Category I

**Problem 1**

*(a) Is there a polynomial $P(x)$ with real coefficients such that*

$$P\left(\frac{1}{k}\right) = \frac{k+2}{k},$$

*for all positive integers $k$?*

*(b) Is there a polynomial $P(x)$ with real coefficients such that*

$$P\left(\frac{1}{k}\right) = \frac{1}{2k+1},$$

*for all positive integers $k$?*

**Solution** $(a)$ YES. It suffices to define a polynomial $W(x)$ as follows

$$W(x) = 2x + 1.$$

$(b)$ NO. Suppose that such a polynomial $W(x)$ exists. Define a polynomial $F(x)$ as follows

$$F(x) = (x+2)W(x) - x.$$

Then

$$F\left(\frac{1}{k}\right) = \left(\frac{1}{k} + 2\right) W\left(\frac{1}{k}\right) - \frac{1}{k} = 0,$$

for all $k \in \mathbb{N}$. Hence, the polynomial $F(x)$ admits infinitely many zeros. Consequently,

$$(x+2)W(x) - x = 0,$$

for all $x \in \mathbb{R}$. But this implies that

$$W(x) = \frac{x}{x+2},$$

for all $x \in \mathbb{R}$ – a contradiction. $\square$

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**Problem 2** *Let $(a_n)_{n=1}^{\infty}$ be unbounded and strictly increasing sequence of positive reals such that the arithmetic mean of any four consecutive terms $a_n$, $a_{n+1}$, $a_{n+2}$, $a_{n+3}$ belongs to the same sequence. Prove that the sequence $a_{n+1}/a_n$ converges and find all possible values of its limit.*

**Solution** Since $a_n < a_{n+1} < a_{n+2} < a_{n+3}$, one has

$$a_n < \frac{1}{4}(a_n + a_{n+1} + a_{n+2} + a_{n+3}) < a_{n+3},$$

thus $(a_n + a_{n+1} + a_{n+2} + a_{n+3})/4 \in \{a_{n+1}, a_{n+2}\}$. Hence for any $n \in \mathbb{N}$ precisely one of the two identities

$$a_n + a_{n+1} + a_{n+2} + a_{n+3} = 4a_{n+1} \tag{1}$$

or

$$a_n + a_{n+1} + a_{n+2} + a_{n+3} = 4a_{n+2} \tag{2}$$

holds. Let $A$ be the set of indices $n \in \mathbb{N}$ for which (1) holds and let $B$ be the set of indices $n \in \mathbb{N}$ for which (2) holds. Clearly, $A \cup B = \mathbb{N}$, $A \cap B = \emptyset$. We shall prove that one of $A$ or $B$ is finite. Indeed, suppose the contrary, that both $A$ and $B$ are infinite. Since $A$ and $B$ partition $\mathbb{N}$, there exists a positive integer $k$, such that $k \in B$, $k+1 \in A$. From (1) and (2), it follows that

$$a_k + a_{k+1} + a_{k+2} + a_{k+3} = 4a_{k+2} \qquad \text{and} \qquad a_{k+1} + a_{k+2} + a_{k+3} + a_{k+4} = 4a_{k+2}.$$

Hence $a_k = a_{k+4}$, which contradicts the fact that $a_n$ is strictly increasing. We now consider two cases.

*Case 1)* The set $A$ is infinite, the set $B$ is finite. By (1), the sequence $a_n$ satisfies a linear recurrence $a_n - 3a_{n+1} + a_{n+2} + a_{n+3} = 0$ for all $n > n_0$. The characteristic polynomial of the linear recurrence

$$\phi(\lambda) = \lambda^3 + \lambda^2 - 3\lambda + 1 = (\lambda - 1)(\lambda^2 + 2\lambda - 1)$$

has roots $\lambda_1 = 1$, $\lambda_2 = -1 - \sqrt{2}$, $\lambda_3 = -1 + \sqrt{2}$. Hence

$$a_n = C_1 + C_2(-1 - \sqrt{2})^n + C_3(-1 + \sqrt{2})^n, \qquad C_1, C_2, C_3 \in \mathbb{R}, \qquad n > n_0.$$

Observe that $\lambda_2 < -1$, $0 < \lambda_3 < 1$. If $C_2 \neq 0$, then $\lim_{n\to\infty} |a_n| = \infty$ and $a_n$ alternates in sign for $n$ sufficiently large which contradicts the monotonicity property. If $C_2 = 0$, then the sequence $a_n$ is bounded, which leads to the contradiction again. Thus we reject the case one.

*Case 2)* The set $A$ is finite, the set $B$ is infinite. By (1), the sequence $a_n$ satisfies a linear recurrence $a_n + a_{n+1} - 3a_{n+2} + a_{n+3} = 0$ for all $n > n_0$. The characteristic polynomial of the linear recurrence

$$\phi(\lambda) = \lambda^3 - 3\lambda^2 + \lambda + 1 = (\lambda - 1)(\lambda^2 - 2\lambda - 1)$$

has roots $\lambda_1 = 1$, $\lambda_2 = 1 - \sqrt{2}$, $\lambda_3 = 1 + \sqrt{2}$. Hence

$$a_n = C_1 + C_2(1 - \sqrt{2})^n + C_3(1 + \sqrt{2})^n, \qquad C_1, C_2, C_3 \in \mathbb{R}, \qquad n > n_0.$$

Note that $-1 < \lambda_2 < 0$, $\lambda_3 > 1$. If $C_3 \leq 0$, then the sequence $a_n$ is bounded from above. Hence $C_3 > 0$ so $a_n \sim C_3\lambda_3^n$ as $n \to \infty$ . The standard limit calculation now shows that $b_n$ converges and has limit value

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda_3 = 1 + \sqrt{2}.$$

□

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**Problem 3** *Prove that*

$$\sum_{k=0}^{\infty} x^k \frac{1+x^{2k+2}}{(1-x^{2k+2})^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(1-x^{k+1})^2}$$

*for all* $x \in (-1,1)$.

**Solution** We use the binomial series

$$\frac{1}{(1-u)^2} = \sum_{j=0}^{\infty} (j+1) u^j, \ |u| < 1$$

to get

$$\sum_{k=0}^{\infty} x^k \frac{1+x^{2k+2}}{(1-x^{2k+2})^2} = \sum_{k=0}^{\infty} x^k (1+x^{2k+2}) \sum_{j=0}^{\infty} (j+1) x^{j(2k+2)} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} x^k (1+x^{2k+2})(j+1) x^{j(2k+2)} =$$

$$= \sum_{j=0}^{\infty} (j+1) x^{2j} \sum_{k=0}^{\infty} x^k (1+x^{2k+2}) x^{j2k} = \sum_{j=0}^{\infty} (j+1) x^{2j} \left( \frac{1}{1-x^{2j+1}} + \frac{x^2}{1-x^{2j+3}} \right) =$$

$$= \sum_{j=0}^{\infty} \frac{(j+1)x^{2j}}{1-x^{2j+1}} + \sum_{j=1}^{\infty} \frac{j x^{2j}}{1-x^{2j+1}} = \sum_{j=0}^{\infty} \frac{(2j+1)x^{2j}}{1-x^{2j+1}} = -\frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=0}^{\infty} \log(1-x^{2j+1})$$

and

$$\sum_{k=0}^{\infty} \frac{(-x)^k}{(1-x^{k+1})^2} = \sum_{k=0}^{\infty} (-x)^k \sum_{j=0}^{\infty} (j+1) x^{(k+1)j} = \sum_{j=0}^{\infty} (j+1) x^j \sum_{k=0}^{\infty} (-x)^k x^{kj} = \sum_{j=0}^{\infty} \frac{(j+1)x^j}{1+x^{j+1}} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{j=0}^{\infty} \log(1+x^{j+1}).$$

The proposition now follows by logarithmic differentiation of the classical identity

$$\prod_{n=0}^{\infty} \frac{1}{1-x^{2n+1}} = \prod_{n=1}^{\infty} (1+x^n),$$

which can be proved as follows:

$$\prod_{n=1}^{\infty} (1+x^n) = \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^n} = \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^n)} = \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^{2n}) \prod_{n=1}^{\infty}(1-x^{2n-1})} = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}.$$

□

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**Problem 4** *Let $a, b, c$ be elements of finite order in some group. Prove that if $a^{-1}ba = b^2$, $b^{-2}cb^2 = c^2$ and $c^{-3}ac^3 = a^2$, then $a = b = c = e$, where $e$ is the unit element.*

**Solution** Let $r(g)$ denote the rank of $g \in G$. Assume that the assertion does not hold. Let $p$ be the smallest prime number dividing $r(a)r(b)r(c)$. Without loss of generality we can assume that $p \mid r(b)$ (if $p \mid r(a)$ or $p \mid r(c)$, then the reasoning is the same). Then there exists $k$ such that $r(b) = pk$. Let $d := b^k$. Then $r(d) = p$.

**Lemma** *For any $m \in \mathbb{N}$, $a^{-m}da^m = d^{2^m}$.*

**Proof** First we prove that

$$a^{-1}da = d^2.$$

Indeed, multiplying the equation $a^{-1}ba = b^2$ k-times with itself we get

$$(a^{-1}ba)(a^{-1}ba)\cdots(a^{-1}ba) = b^2b^2\cdots b^2;$$

and hence

$$a^{-1}b^ka = (b^2)^k = (b^k)^2.$$

Now, the assertion of the above lemma follows from the following calculations:

$$d = ad^2a^{-1} = a(ad^2a^{-1})^2a^{-1} = a^2d^{2^2}a^{-2} = a^2(ad^2a^{-1})^{2^2}a^{-2} = a^3d^{2^3}a^{-3} = \cdots = a^md^{2^m}a^{-m}. \tag{1}$$

□

Observe that Fermat's little theorem implies that $2^p \equiv 2 \pmod{p}$. Consequently,

$$a^{-p}da^p = d^{2^p} = d^2 = a^{-1}da. \tag{2}$$

Since $\gcd(r(a), p-1) = 1$, there exist integers r and s such that

$$r \cdot r(a) + s \cdot (p-1) = 1. \tag{3}$$

From (2) we get

$$a^{-l(p-1)}da^{l(p-1)} = d,$$

for all $l \in \mathbb{Z}$ (see the calculations in (1)). Finally, putting $l := s$, we obtain

$$d = a^{-s(p-1)}da^{s(p-1)} \overset{(3)}{=} a^{rr(a)-1}da^{-rr(a)+1} = a^{-1}da = d^2,$$

which implies that $d = e$, a contradiction. □