The 21st Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31st March 2011
Category I

Problem 1

(a) *Is there a polynomial* $P(x)$ *with real coefficients such that*

$$P\left(\frac{1}{k}\right) = \frac{k+2}{k}$$

for all positive integers k*?*

(b) *Is there a polynomial* $P(x)$ *with real coefficients such that*

$$P\left(\frac{1}{k}\right) = \frac{1}{2k+1}$$

for all positive integers k*?*

Problem 2 *Let* $(a_n)_{n=1}^{\infty}$ *be an unbounded and strictly increasing sequence of positive reals such that the arithmetic mean of any four consecutive terms* a_n, a_{n+1}, a_{n+2}, a_{n+3} *belongs to the same sequence. Prove that the sequence* a_{n+1}/a_n *converges and find all possible values of its limit.*

Problem 3 *Prove that*

$$\sum_{k=0}^{\infty} x^k \frac{1+x^{2k+2}}{(1-x^{2k+2})^2} = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{(1-x^{k+1})^2}$$

for all $x \in (-1,1)$.

Problem 4 *Let* a, b, c *be elements of finite order in some group. Prove that if* $a^{-1}ba = b^2$, $b^{-2}cb^2 = c^2$ *and* $c^{-3}ac^3 = a^2$ *then* $a = b = c = e$*, where* e *is the unit element.*