The 20th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 25th March 2010
Category II

Problem 1 *Let a and b be given positive coprime integers. Then for every integer n there exist integers x, y such that*

$$n = ax + by\,.$$

Prove that $n = ab$ is the greatest integer for which $xy \le 0$ in all such representations of n. [10 points]

Problem 2 *Prove or disprove that if a real sequence (a_n) satisfies $a_{n+1} - a_n \to 0$ and $a_{2n} - 2a_n \to 0$ as $n \to \infty$, then $a_n \to 0$.* [10 points]

Problem 3 *Let A and B be two $n \times n$ matrices with integer entries such that all of the matrices*

$$A\,, \quad A+B\,, \quad A+2B\,, \quad A+3B\,, \quad \dots\,, \quad A+(2n)B$$

are invertible and their inverses have integer entries, too. Show that $A+(2n+1)B$ is also invertible and that its inverse has integer entries. [10 points]

Problem 4 *Let $f\colon [0,1] \to \mathbb{R}$ be a function satisfying*

$$|f(x) - f(y)| \le |x - y|$$

for every $x, y \in [0,1]$. Show that for every $\varepsilon > 0$ there exists a countable family of rectangles (R_i) of dimensions $a_i \times b_i$, $a_i \le b_i$, in the plane such that

$$\big\{(x, f(x)) : x \in [0,1]\big\} \subset \bigcup_i R_i \quad \text{and} \quad \sum_i a_i < \varepsilon\,.$$

(The edges of the rectangles are not necessarily parallel to the coordinate axes.) [10 points]