The 20^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 25^{th} March 2010
Category I

Problem 1

a) Is it true that for every bijection $f\colon \mathbb{N} \to \mathbb{N}$ the series

$$\sum_{n=1}^{\infty} \frac{1}{nf(n)}$$

is convergent?

b) Prove that there exists a bijection $f\colon \mathbb{N} \to \mathbb{N}$ such that the series

$$\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$$

is convergent.

($\mathbb{N}$ is the set of all positive integers.) [10 points]

Problem 2 *Let A and B be two complex 2×2 matrices such that $AB - BA = B^2$. Prove that $AB = BA$.* [10 points]

Problem 3 *Prove that there exist positive constants c_1 and c_2 with the following properties:*

a) For all real $k > 1$,

$$\left| \int_0^1 \sqrt{1-x^2}\,\cos(kx)\,\mathrm{d}x \right| < \frac{c_1}{k^{3/2}}\,.$$

b) For all real $k > 1$,

$$\left| \int_0^1 \sqrt{1-x^2}\,\sin(kx)\,\mathrm{d}x \right| > \frac{c_2}{k}\,.$$

[10 points]

Problem 4 *For every positive integer n let $\sigma(n)$ denote the sum of all its positive divisors. A number n is called* weird *if $\sigma(n) \geq 2n$ and there exists no representation*

$$n = d_1 + d_2 + \cdots + d_r\,,$$

where $r > 1$ and $d_1, \ldots, d_r$ are pairwise distinct positive divisors of n.
Prove that there are infinitely many weird numbers. [10 points]