The 19$^{\text{th}}$ Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 1$^{\text{st}}$ April 2009
Category I

Problem 1 *Let ABC be a non-degenerate triangle in the euclidean plane. Define a sequence $(C_n)_{n=0}^{\infty}$ of points as follows: $C_0 := C$, and C_{n+1} is the center of the incircle of the triangle ABC_n. Find $\lim_{n\to\infty} C_n$.*

[10 points]

Solution If α is the angle at A, β the angle at B, then the limit is the point on the side $\overline{AB}$ dividing it in the ratio $\alpha : \beta$. Let α_i and β_i be the angles at A and B in ABC_i, respectively. Since the center of the incircle is the intersection of the angle bisectors, we have $\alpha_{i+1} = \frac{\alpha_i}{2}$ and $\beta_{i+1} = \frac{\beta_i}{2}$; so the limit point will obviously lie on $\overline{AB}$; furthermore, $\frac{\alpha_i}{\beta_i} = \frac{\alpha}{\beta} =: q$ for all i. Thus, if K_i is the circumcircle of ABC_i, $S_{1,i}$ and $S_{2,i}$ the arcs over $\overline{AC_i}$ and $\overline{BC_i}$, respectively, then $\frac{|S_{1,i}|}{|S_{2,i}|} = q$ for all i. Now, as the C_i approache $\overline{AB}$, the arcs converge to the corresponding sides of the triangle. Hence, the result follows. $\square$

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Problem 2 *Prove that the number*

$$2^{2^k - 1} - 2^k - 1$$

is composite (not prime) for all positive integers $k > 2$. [10 points]

Solution Denote

$$M = 2^{2^k-1} - 2^k - 1\,.$$

If k is even then $3 \mid M$, and M is composite, since $M > 3$ for $k > 2$.

Suppose k is odd. Then

$$2M = 2^{2^k} - 1 - \left(2^{k+1} + 1\right) = \left(2^{2^{k-1}} + 1\right)\left(2^{2^{k-2}} + 1\right)\dots\left(2^{2^1} + 1\right) - \left(2^{k+1} + 1\right)\,.$$

Let $k + 1 = 2^a q$ with positive odd integer q and $a \geq 1$. Then $\left(2^{2^a} + 1\right) \mid 2M$. Indeed, $\left(2^{2^a} + 1\right) \mid \left(2^{k+1} + 1\right)$ and

$$\left(2^{2^a} + 1\right) \,\Big|\, \left(2^{2^{k-1}} + 1\right)\left(2^{2^{k-2}} + 1\right)\dots\left(2^{2^1} + 1\right)\,,$$

since $a \leq k - 1$ for $k > 2$. $\square$

Problem 3 *Let k and n be positive integers such that $k \leq n-1$. Let $S := \{1, 2, \ldots, n\}$ and let $A_1, A_2, \ldots, A_k$ be nonempty subsets of S. Prove that it is possible to color some elements of S using two colors, red and blue, such that the following conditions are satisfied:*

(i) Each element of S is either left uncolored or is colored red or blue.

(ii) At least one element of S is colored.

(iii) Each set A_i ($i = 1, 2, \ldots, k$) is either completely uncolored or it contains at least one red and at least one blue element.

[10 points]

Solution Consider the following system of k linear equations in n real variables $x_1, x_2, \ldots, x_n$:

$$\sum_{j \in A_i} x_j = 0\,, \quad i = 1, 2, \ldots, k\,.$$

Since $k < n$, this system has a nontrivial solution $(x_1, x_2, \ldots, x_n)$, i.e. a solution with at least one nonzero x_j. Now color red all elements of the set $\{j \in S : x_j > 0\}$, color blue all elements of the set $\{j \in S : x_j < 0\}$, and leave uncolored all elements of $\{j \in S : x_j = 0\}$.

Since the solution is nontrivial, at least one element is colored. If A_i contains some red element $j \in S$ then $x_j > 0$, and from $\sum_{j \in A_i} x_j = 0$ we see that there exists some $j' \in A_i$ such that $x_{j'} < 0$, i.e. j' is colored blue. Thus A_i must have elements of both colors. Analogously we argue when A_i contains a blue element. Therefore we see that the above coloring satisfies all requirements. $\square$

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Problem 4 *Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. We say that the sequence $(a_n)_{n=1}^{\infty}$* covers *the set of positive integers if for any positive integer m there exists a positive integer k such that $\sum\limits_{n=1}^{\infty} a_n^k = m$.*

a) Does there exist a sequence of real positive numbers which covers the set of positive integers?

b) Does there exist a sequence of real numbers which covers the set of positive integers?

[10 points]

Solution The answer to the second question is positive.

First we shall prove that for any n there exists a finite sequence $(x_i)_{i=1}^{k_n}$ of real numbers such that

$$\sum_{i=1}^{k_n} x_i^{2m+1} = 0 \quad \text{for } 0 \leq m < n$$

and

$$\sum_{i=1}^{k_n} x_i^{2m+1} \neq 0 \quad \text{for } m \geq n\,.$$

For the simplicity of notation we shall write $S_m(x_i)$ for $\sum_{i=1}^{k_n} x_i^{2m+1}$. We shall prove the thesis by induction upon n. For $n = 0$ the appropriate sequence is $x_1 = 1$.

Assume the thesis for n. For $n+1$ consider the sequence

$$(y_i)_{i=1}^{3k_n} = (-x_1, -x_2, \ldots, -x_{k_n}, \alpha x_1, \alpha x_2, \ldots, \alpha x_{k_n}, \alpha x_1, \alpha x_2, \ldots, \alpha x_{k_n})\,,$$

where $\alpha = 2^{-1/(2n+1)}$. As $S_m(x_i) = 0$ for $m < n$, we also have $S_m(y_i) = 0$. We also have

$$S_n(y_i) = -S_n(x_i) + 2^{-1}S_n(x_i) + 2^{-1}S_n(x_i) = 0\,.$$

For $m > n$ we have

$$S_m(y_i) = (1 - 2 \cdot 2^{-(2m+1)/(2n+1)})S_m(x_i) \neq 0\,.$$

Thus the induction step is finished, and the thesis is proved. Moreover it is easy to notice that $|x_i| \leq 1$ and the length of the sequence is 3^n. Denote by $x(n)$ the sequence of length 3^n with $S_m(x(n)) = 0$ for $m < n$.

Now to give the required sequence (a_i). Our sequence will be a concatenation of multiples of the finite sequences $x(n)$ given above. We begin with $a_1 = 1$ (that is we begin by taking $x(0)$). In the n-th step we assume that we have some finite sequence a_i, with $S_m(a_i) = m+1$ for $m \leq n$. We also assume that the elements added in the n-th step will be no larger than $\frac{1}{n}$.

To pass to the $(n+1)$-st step let $c = n + 2 - S_{n+1}(a_i)$, and let $d = S_{n+1}(x(n+1))$. Take an integer $N > \left|\frac{(n+1)c}{d}\right|$, let $\alpha = \frac{c}{dN}$, $|\alpha| < \frac{1}{n+1}$. We add N copies of the sequence $\alpha x(n+1)$ to the end of a_i. This does not change $S_m(a_i)$ for $m < n+1$ (as $S_m(x(n+1)) = 0$, and after the addition we have $S_{n+1}(a_i) = n+2$. Also all the added elements are of absolute value no larger than $\frac{1}{n+1}$.

Now to prove that for this series we have $G_{2k+1} = k+1$. As $S_m(a_i) = m+1$ after every step, no other limit is possible, we only have to check convergence. Note, however, that after the n–th step we only add sequences $x(m)$ for $m > n$, which in turn are concatenations of sequences $x(n)$, with some coefficients. Thus every 3^n–th partial sum in the series $\sum a_i^{2n+1}$ is going to be exactly equal to $n+1$. The partial sums "in the middle" cannot differ from this value by more than $\frac{3^n}{2}$ times the value of the maximal element $|a_i|$ in the appropriate interval, and this converges to zero. Thus for any n we do, in fact, have convergence.

For the first question, obviously the same series suffices.

For the last question, the answer is negative. As a_i are positive, we may rearrange them in decreasing order. Take k_0 to be the first k for which G_k is finite. For G_{k_0} to be finite, we have to have a_i convergent to zero, thus only a finite number of terms is larger than 1, assume these are the first n terms. Note that as for $i > n$ we have $a_i \leq 1$, we also have that a_i^k decreases with k, and thus $\sum_{i=n+1}^{\infty} a_i^k$ decreases with k, and thus is bounded by $C := \sum_{i=n+1}^{\infty} a_i^{k_0}$. As G_k are assumed to attain unbounded values, we have to have terms larger than 1, thus $n > 0$.

Assume the first m terms of a_i are equal, $1 \leq m \leq n$. Then for $k \geq k_0$ and $l \leq k$ we have

$$G_l \leq m a_1^k + n a_{m+1}^k + C\,.$$

On the other hand $G_l \geq ma_1^{k+1}$ for $l \geq k$. For sufficiently large k, however, we have

$$ma_1^{k+1} > ma_1^k + na_{m+1}^k + C + 2\,,$$

which means that there is an integer number between ma_1^{k+1} and $ma_1^k + na_{m+1}^k + C$ which is not the value of G_k for any k. $\square$