Problem j18-II-1. *Find all functions* $f\colon \mathbb{Z} \to \mathbb{Z}$ *such that*

$$19f(x) - 17f(f(x)) = 2x \tag{1}$$

for all $x \in \mathbb{Z}$.

Solution. Suppose that there exists a function $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying the above equation. Then define a function $g\colon \mathbb{Z} \to \mathbb{Z}$ by

$$g(x) = x - f(x)\,. \tag{2}$$

Taking into account (1) and (2), we get

$$17g(f(x)) = 2g(x)\,. \tag{3}$$

Let us fix $y \in \mathbb{Z}$ and let $a := g(y)$. Define a sequence $(x_n)_{n\geq 0}$ as follows

$$x_0 := y\,, \quad x_1 := f(x_0)\,, \quad \ldots, \quad x_n := f(x_{n-1})\,, \quad \ldots$$

for any $n \in \mathbb{N}$. Now substituting x_n into (3) in turn, we get

$$a = g(x_0) = \frac{17}{2}g(x_1) = \ldots = \frac{17^n}{2^n}g(x_n)$$

for any $n > 0$. Consequently, we infer that

$$2^n a = 17^n g(x_n)$$

for any $n > 0$. Since 2 and 17 are relatively prime, we deduce that $17^n \mid a$ for any $n > 0$ and therefore $a = 0$. Moreover, since y was arbitrary, it follows that $g(y) = 0$ for any $y \in \mathbb{Z}$. Thus $y - f(y) = 0$ for any $y \in \mathbb{Z}$ and hence $f(y) = y$ for any $y \in \mathbb{Z}$. This implies that only one function satisfies the equation (1). So, this completes the solution. □

Problem j18-II-2. *Find all continuously differentiable functions* $f\colon [0,1] \to (0,\infty)$ *such that* $\frac{f(1)}{f(0)} = \mathrm{e}$ *and*

$$\int_0^1 \frac{\mathrm{d}x}{f(x)^2} + \int_0^1 f'(x)^2\,\mathrm{d}x \le 2\,.$$

Solution. First, we note that if f is such function, then

$$\begin{aligned} 0 &\le \int_0^1 \Big(f'(x) - \frac{1}{f(x)}\Big)^2\,\mathrm{d}x = \int_0^1 f'(x)^2\,\mathrm{d}x - 2\int_0^1 \frac{f'(x)}{f(x)}\,\mathrm{d}x + \int_0^1 \frac{\mathrm{d}x}{f(x)^2} \\ &= \int_0^1 f'(x)^2\,\mathrm{d}x - 2\int_0^1 (\ln f(x))'\,\mathrm{d}x + \int_0^1 \frac{\mathrm{d}x}{f(x)^2} \\ &= \int_0^1 f'(x)^2\,\mathrm{d}x - 2\ln\frac{f(1)}{f(0)} + \int_0^1 \frac{\mathrm{d}x}{f(x)^2}\,\mathrm{d}x \le 0\,, \end{aligned}$$

since $\frac{f(1)}{f(0)} = \mathrm{e}$ and $\int_0^1 \frac{\mathrm{d}x}{f(x)^2} + \int_0^1 f'(x)^2\,\mathrm{d}x \le 2$. Therefore

$$\int_0^1 \Big(f'(x) - \frac{1}{f(x)}\Big)^2\,\mathrm{d}x = 0\,. \tag{1}$$

Since f is continuously differentiable function on $[0,1]$, the equality (1) is equivalent to

$$f'(x)f(x) = 1 \quad \forall x \in [0,1]\,. \tag{2}$$

All positive solutions of the differential equation (2) are in the form $f(x) = \sqrt{2x+C}$ for some $C > 0$. Since $\frac{f(1)}{f(0)} = \mathrm{e}$, we have $C = \frac{2}{\mathrm{e}^2-1}$, and thus

$$f(x) = \sqrt{2x + \frac{2}{\mathrm{e}^2-1}}$$

is the unique function satisfying the conditions from the statement. □

Problem j18-II-3. *Find all pairs of natural numbers* (n, m) *with* $1 < n < m$ *such that the numbers* 1, $\sqrt[n]{n}$ *and* $\sqrt[m]{m}$ *are* linearly dependent *over the field of rational numbers* $\mathbb{Q}$.

Solution. The answer is $n = 2, m = 4$.

We begin with the following

Lemma. *The minimal (over* $\mathbb{Q}$*) polynomial* $f(X)$ *for* $\sqrt[n]{n}$ *equals* $X^k - (\sqrt[n]{n})^k$*, where* k *is the minimal satisfying* $(\sqrt[n]{n})^k \in \mathbb{N}$.

Proof. $\sqrt[n]{n}$ is a root of $X^n - n = 0$. So there is some nonempty subset A of $\{0, 1, ..., n-1\}$ such that

$$f(X) = \prod_{l \in A} (X - \zeta^l)\,,$$

where $\zeta = \cos\left(\frac{2\pi}{n}\right) + \mathrm{i}\sin\left(\frac{2\pi}{n}\right)$.

The free term of $f(X)$ has an absolute value equal to $(\sqrt[n]{n})^{|A|}$. Hence $(\sqrt[n]{n})^{\deg f(X)}$ is integer, and $\deg f(X) \geq k$ follows (k is as in the lemma). But, clearly $\sqrt[n]{n}$ is a root of $X^k - (\sqrt[n]{n})^k$, which has integer coefficients. ▫

Let us assume that 1, $\sqrt[n]{n}$, $\sqrt[m]{m}$ are linearly dependent over $\mathbb{Q}$, i.e. there are rational a, b, c not all equal 0 such that $a + b\sqrt[n]{n} + c\sqrt[m]{m} = 0$.

Case $a \neq 0$. Then, as $\sqrt[n]{n}$ is irrational, we have $b, c \neq 0$. But $a + b\sqrt[n]{n} = -c\sqrt[m]{m}$ has the same degree of a minimal polynomial as $\sqrt[n]{n}$, and as $\sqrt[m]{m}$. Let k be the degree of the minimal polynomial for $\sqrt[m]{m}$. Then $y = \sqrt[n]{n}$ satisfies

$$(a + by)^k = (\sqrt[m]{m})^k\,,$$

but y^k and $(\sqrt[m]{m})^k$ are rational, and as $a, b \neq 0$ we obtain that there is a nonzero polynomial with rational coefficients vanishing $\sqrt[n]{n}$ of degree smaller than k, a contradiction.

Case $a = 0$. Hence $\frac{\sqrt[n]{n}}{\sqrt[m]{m}}$ is rational, and this is equivalent to $\frac{n^m}{m^n}$ is a mn-th power of a rational. Let p be any prime, and $p^a \parallel n$, $p^b \parallel m$. So we must have $mn \mid am - bn$. But $am - bn \leq am < mn$, in view of $a \leq \log_2 n < n$. In a similar way one obtains $am - bn > -mn$. So we must have $am = bn$, the relation independent of the choice of prime p. Thus

$$n = m^{m/n},$$

and $\sqrt[n]{n} = \sqrt[m]{m}$ follows. As the function $\sqrt[x]{x}$ has maximum at $x = \mathrm{e}$, we see that $\sqrt[n]{n} = \sqrt[m]{m}$ holds only for $n = 2, m = 4$. □

Problem j18-II-4. *We consider the following game for one person. The aim of the player is to reach a fixed capital $C > 2$. The player begins with capital $0 < x_0 < C$. In each turn let x be the player's current capital. Define $s(x)$ as follows:*

$$s(x) := \begin{cases} x & \text{if } x < 1 \\ C - x & \text{if } C - x < 1 \\ 1 & \text{otherwise.} \end{cases}$$

Then a fair coin is tossed and the player's capital either increases or decreases by $s(x)$, each with probability $\frac{1}{2}$. Find the probability that in a finite number of turns the player wins by reaching the capital C.

Solution. Let us denote by $f(x)$ the probability that player wins with starting capital x.

If $x \le 1$, then he loses if loses the first turn, and if he wins the first turn, he has capital $2x$. Thus $f(x) = \frac{1}{2}f(2x)$.

If $x \ge C - 1$ the player wins if he wins the first turn, and has $2x - C$ in other case, thus $f(x) = \frac{1}{2} + \frac{1}{2}f(2x - C)$.

In all other cases there is $f(x) = \frac{1}{2}\big(f(x-1) + f(x+1)\big)$.

We will prove that this implies $f(x) = \frac{x}{C}$.

Let us define $g(x) = f(x) - \frac{x}{C}$. It is bounded on $[0, C]$ (as $f(x) \in [0, 1]$), and we have

$$g(x) = \begin{cases} \frac{1}{2}f(2x) - \frac{x}{C} = \frac{1}{2}\big(f(2x) - \frac{2x}{C}\big) = \frac{1}{2}g(2x) & \text{for } x \le 1, \\ \frac{1}{2}\big(f(x-1) + f(x+1)\big) - \frac{x}{C} & \\ \quad = \frac{1}{2}\big(f(x-1) - \frac{x-1}{C} + f(x+1) - \frac{x+1}{C}\big) & \\ \quad = \frac{1}{2}\big(g(x-1) + g(x+1)\big) & \text{for } x \in (1, C-1), \\ \frac{1}{2} + \frac{1}{2}f(2x - C) - \frac{x}{C} & \\ \quad = \frac{1}{2}\big(f(2x - C) - \frac{2x-C}{C}\big) = \frac{1}{2}g(2x - C) & \text{for } x \ge C - 1. \end{cases}$$

Obviously $g(0) = g(C) = 0$. Let $K = \sup\limits_{t \in [0,C]} f(t) \in [0, \infty)$. Denote $n_0 = [C] - 1 \ge 1$.

We will prove for any natural $0 < n \le n_0$ and $x \in (n-1, n]$ there is $g(x) \le \frac{2^n - 1}{2^n}K$.

If $x \in (0, 1]$ there is $g(x) = \frac{1}{2}g(2x) \le \frac{K}{2}$.

Assume, that for $x \le n - 1$ and take $\bar{x} \in (n-1, n]$. There is $g(\bar{x} - 1) \le \frac{2^{n-1}-1}{2^{n-1}}K$ as $\bar{x} - 1 \in (n-2, n-1]$, and $g(\bar{x} + 1) \le K$. Thus

$$g(\bar{x}) = \frac{1}{2}\big(g(\bar{x} - 1) + g(\bar{x} + 1)\big) \le \frac{1}{2}\Big(\frac{2^{n-1} - 1}{2^{n-1}}K + K\Big) = \frac{2^n - 1}{2^n}K$$

as required.

$g(x) \le= \frac{1}{2}g(2x - C) \le \frac{K}{2}$ for $x \ge C - 1$.

Now take $x \in (n_0, C-1)$ (it is empty set for integer C). We have proved that $g(x-1) \le \frac{2^{n_0}-1}{2^{n_0}}K$ (as $x - 1 \in (n_0 - 1, n_0)$) and $g(x+1) \le \frac{K}{2}$ $(x + 1 > C - 1)$. Thus $g(x) \le \frac{2^{n_0}-1}{2^{n_0}}K$.

Thus we have proved, that $g(x) \le \frac{2^{n_0}-1}{2^{n_0}}K$ for every $x \in [0, C]$, which means that $K = 0$. Similarly one can prove, that $\inf\limits_{t \in [0,C]} f(t) = 0$. Thus $g(x) \equiv 0$, so $f(x) = \frac{x}{C}$. □