Category I

Problem 1. *Can the set of positive rationals be split into two nonempty disjoint subsets* Q_1 *and* Q_2*, such that both are closed under addition, i.e.* $p + q \in Q_k$ *for every* $p, q \in Q_k$*,* $k = 1, 2$*? Can it be done when addition is exchanged for multiplication, i.e.* $p \cdot q \in Q_k$ *for every* $p, q \in Q_k$*,* $k = 1, 2$*?*

Solution. (a) No. If $\frac{p}{q}, \frac{r}{s} \in Q_k$ then of course $\frac{ps+qr}{qs} \in Q_k$. Adding n times $\frac{p}{q}$ and m times $\frac{r}{s}$ gives $\frac{nps+mqr}{qs} \in Q_k$ for all positive integers n, m, hence $\tilde{n}p + \tilde{m}r \in Q_k$ for all positive integers $\tilde{n}, \tilde{m}$. So if $\frac{p_k}{q_k}, \frac{r_k}{s_k} \in Q_k$ we get that $p_1p_2 + r_1r_2 \in Q_1 \cap Q_2$.

(b) Yes, for instance

$$Q_1 = \left\{ \frac{m}{n} \in \mathbb{Q}^+ : (m, n) = 1 \text{ and } 2 \mid n \right\} \quad \text{and} \quad Q_2 = \mathbb{Q}^+ \setminus Q_1 .$$

$\square$

Problem 2. *Alice has got a circular key ring with n keys, $n \geq 3$. When she takes it out of her pocket, she does not know whether it got rotated and/or flipped. The only way she can distinguish the keys is by colouring them (a colour is assigned to each key). What is the minimum number of colors needed?*

Solution. Clearly at least two colors are needed in any case to distinguish between at least two keys. For three, four or five keys on the ring, we will show that three colors are necessary. For six or more keys on the ring, we will show that two colors suffice. Choose one key and denote it with k_1. Order all other keys in natural order as they follow each other going from k_1 around the ring in one direction. For $1 \leq i \leq n$ denote with $c(k_i)$ color of the key k_i. Without loss of generality let $c(k_1) = 1$.

Suppose that two colors suffice for $n = 3$. Then there are two similar possibilities for coloring the keys. Either $c(k_2) = c(k_3) = 2$ or $c(k_2) = 1$. In the first case one can not distinguish between keys k_2 and k_3. In the second case one can not distinguish between keys k_1 and k_2. Hence for $n = 3$ we need three colors.

Suppose that two colors suffice for $n = 4$. Then there are four possibilities for coloring the keys. If $c(k_2) = c(k_3) = c(k_4) = 2$, then k_2 and k_4 can not be distinguished (rotation of the key ring through the line across k_1 and k_3 interchanges k_2 and k_4). If $c(k_2) = 1$ and $c(k_3) = c(k_4) = 2$ then there is a rotation that interchanges k_1 and k_2 and also interchanges k_3 and k_4 (similar is the case when $c(k_4) = 1$ and $c(k_2) = c(k_3) = 2$). If $c(k_3) = 1$ and $c(k_2) = c(k_4) = 2$ then there is a rotation that interchanges k_1 and k_3 and there is also other rotation that interchanges k_2 and k_4. Hence for $n = 4$ at least three colors are needed. Consider the following coloring: $c(k_1) = 1$, $c(k_2) = 2$, $c(k_3) = 3$ and $c(k_4) = 1$ (one possibility). Keys k_1 and k_4 have the same color, but one can distinguish between them since k_1 has a neighbor colored with color 1 and a neighbor colored with color 2, while k_4 has also one neighbor colored with color 1, but the other neighbor is colored with color 3. Hence three colors suffice for $n = 4$.

Suppose that two colors suffice for $n = 5$. Then there are two possibilities for coloring the keys: all other keys than k_1 are colored with color 2 (the similar is the case when one key gets color 1, only the roles of the colors are interchanged) or one of them gets color 1 and other three get color 2 (the same is the case when two keys get color 2, only the roles of the colors are interchanged). In first case one can not distinguish between keys k_2 and k_5 and also between keys k_3 and k_4 (there is a rotation of the key ring where keys in both pairs interchange, while k_1 is fixed). When there is a key other than k_1 with color 1 we need to consider two subcases. If $c(k_2) = 1$ (similar is the case when $c(k_5) = 1$) we can not distinguish between k_1 and k_2 (also between k_3 and k_5). If $c(k_3) = 1$ (similar is the case when $c(k_4) = 1$) we can not distinguish between k_1 and k_3 (also between k_4 and k_5). Hence for $n = 5$ at least three colors are needed. Consider the following coloring: $c(k_1) = 1$, $c(k_2) = 2$, $c(k_3) = 3$ and $c(k_4) = c(k_5) = 2$ (one possibility). Keys k_2, k_4 and k_5 have the same color, but one can distinguish between them since k_2 is the only one between them that has a neighbor colored with color 1 and a neighbor colored with color 3, while only k_4 has a neighbor colored with color 3 and a neighbor colored with color 2. Hence three colors suffice for $n = 5$.

For $n \geq 6$ consider the following coloring: $c(k_1) = 1$, $c(k_n) = 2$, $c(k_{n-1}) = c(k_{n-2}) = 1$ and $c(k_i) = 2$ for $2 \leq i \leq n-3$. Then k_1 is the only key of color 1 with both neighbors colored with color 2. Keys k_{n-1} and k_{n-2} both have neighbors of two different colors, but the distance (the smallest of the two numbers: number of the keys lying between the two keys in one and other direction) between k_{n-1} and k_1 is one while the distance between k_{n-2} and k_1 is two. Hence one can distinguish between all three keys colored with color 1. Among keys colored with color 2 only k_n has both neighbors colored with color 1. All other keys: k_i for $2 \leq i \leq n-3$ have either one or two neighbors colored with color 2. But any k_i, where $2 \leq i \leq n-3$, has a pair of distances: distance between k_i and k_1 and distance between k_i and k_{n-2} that is different from any other pair of distances of some key $k_j \neq k_i$ for $2 \leq j \leq n-3$. Hence we can distinguish also between keys colored with color 2. $\square$

Problem 3. *A function* $f\colon [0,\infty) \to \mathbb{R} \setminus \{0\}$ *is called* slowly changing *if for any* $t > 1$ *the limit* $\lim_{x\to\infty} \frac{f(tx)}{f(x)}$ *exists and is equal to 1. Is it true that every slowly changing function has for sufficiently large* x *a constant sign (that is — it is true that for every slowly changing* f *there exists* N *such that for every* $x, y > N$ *we have* $f(x)f(y) > 0$*?)*

Remark. The assumption $f(x) \neq 0$ is only technical, to avoid explaining what does the limit mean in the other case, and in reality changes nothing.

Remark. The reader is encouraged to try and solve the problem himself before reading the solution. The author's and the proposer's opinion is that although the solution is simple, it is not so easy to find it (both tried, both succeeded, but both spent some time on it before getting the correct idea).

Solution. Take $t = 2$. Take such a $N > 0$ that for $x > N$ we have $\frac{f(2x)}{f(x)} > 0$. This means $f(2x)$ and $f(x)$ are of the same sign for $x > N$. Suppose that for any $x > N$ we have that $f(x)$ and $f(N)$ are of a different sign. Let $t = \frac{x}{N}$. Then $\frac{f(tN)}{f(N)} < 0$, and by easy induction $\frac{f(t2^kN)}{f(2^kN)} < 0$ for any $k \in \mathbb{N}$, which contradicts the assumption $\frac{f(tx)}{f(x)} \to 1$ when x tends to ∞. The contradiction proves the thesis. $\square$

Problem 4. *Let $f\colon [0,1] \to [0,\infty)$ be an arbitrary function satisfying*

$$\frac{f(x)+f(y)}{2} \le f\Big(\frac{x+y}{2}\Big) + 1 \tag{1}$$

for all pairs $x, y \in [0,1]$. Prove that for all $1 \le u < v < w \le 1$,

$$\frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) \le f(v) + 2\,.$$

Solution. Let

$$M(u,w) = \sup_{v\in(u,w)} \bigg(\frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) - f(v) \bigg)\,;$$

we have to prove $M(u,w) \le 2$. Note that $M(u,w)$ is finite, because

$$\frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) - f(v) \le 1\cdot f(u) + 1\cdot f(w) - 0 = f(u)+f(w)\,.$$

Let $\varepsilon > 0$ be an arbitrary positive real number. Choose v such that

$$\frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) - f(v) > M(u,w) - \varepsilon\,.$$

If $v \le \frac{u+w}{2}$, then apply (1) for $x = u$ and $y = u + 2(v-u) = 2v-u$:

$$\frac{f(u)+f(2v-u)}{2} \le f(v) + 1\,;$$

$$\begin{aligned}
M(u,w) - \varepsilon &< \frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) - f(v) \\
&\le \frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) - \frac{f(u)+f(2v-u)}{2} + 1 \\
&= \frac{1}{2}\bigg(\frac{w-(2v-u)}{w-u} f(u) + \frac{(2v-u)-u}{w-u} f(w) - f(2v-u) \bigg) + 1 \\
&\le \frac{1}{2} M(u,w) + 1\,;
\end{aligned}$$

$$M(u,w) \le 2 + 2\varepsilon\,.$$

Otherwise, if $\frac{u+w}{2} < v$, apply $x = w - 2(w-v) = 2v-w$ and $y = v$ in (1):

$$\frac{f(2v-w)+f(w)}{2} \le f(v) + 1\,;$$

$$\begin{aligned}
M(u,w) - \varepsilon &< \frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) - f(v) \\
&\le \frac{w-v}{w-u} f(u) + \frac{v-u}{w-u} f(w) - \frac{f(2v-w)+f(w)}{2} + 1 \\
&= \frac{1}{2}\bigg(\frac{w-(2v-w)}{w-u} f(u) + \frac{(2v-w)-u}{w-u} f(w) - f(2v-w) \bigg) + 1 \\
&\le \frac{1}{2} M(u,w) + 1\,;
\end{aligned}$$

$$M(u,w) \le 2 + 2\varepsilon\,.$$

In both cases we obtained $M(u,w) \le 2 + 2\varepsilon$. This holds for all ε, therefore $M(u,w) \le 2$. □