The 17th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 28th March 2007
Category I

Problem 1. *Can the set of positive rationals be split into two nonempty disjoint subsets* Q_1 *and* Q_2*, such that both are closed under addition, i.e.* $p + q \in Q_k$ *for every* $p, q \in Q_k$, $k = 1, 2$*?*

Can it be done when addition is exchanged for multiplication, i.e. $p \cdot q \in Q_k$ *for every* $p, q \in Q_k$, $k = 1, 2$*?*

[10 points]

Problem 2. *Alice has got a circular key ring with* n *keys,* $n \geq 3$*. When she takes it out of her pocket, she does not know whether it got rotated and/or flipped. The only way she can distinguish the keys is by colouring them (a colour is assigned to each key). What is the minimum number of colours needed?* [10 points]

Problem 3. *A function* $f\colon [0, \infty) \to \mathbb{R} \setminus \{0\}$ *is called* slowly changing *if for any* $t > 1$ *the limit* $\lim_{x\to\infty} \frac{f(tx)}{f(x)}$ *exists and is equal to* 1*. Is it true that every slowly changing function has for sufficiently large* x *a constant sign (i.e., is it true that for every slowly changing* f *there exists an* N *such that for every* $x, y > N$ *we have* $f(x)f(y) > 0$*?)* [10 points]

Problem 4. *Let* $f\colon [0, 1] \to [0, \infty)$ *be an arbitrary function satisfying*

$$\frac{f(x) + f(y)}{2} \leq f\Big(\frac{x + y}{2}\Big) + 1$$

for all pairs $x, y \in [0, 1]$*. Prove that for all* $0 \leq u < v < w \leq 1$,

$$\frac{w - v}{w - u} f(u) + \frac{v - u}{w - u} f(w) \leq f(v) + 2\,.$$

[10 points]