The 16^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 29^{th} March 2006
Category I

Problem 1 *Given real numbers* $0 = x_1 < x_2 < \cdots < x_{2n} < x_{2n+1} = 1$ *such that* $x_{i+1} - x_i \le h$ *for* $1 \le i \le 2n$, *show that*

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) < \frac{1+h}{2}\,.$$

Solution (by Stijn Cambie) Notice that $\sum_{i=1}^{n}(x_{2i+1} + x_{2i-1})(x_{2i+1} - x_{2i-1}) = x_{2n+1}^2 - x_1^2 = 1$. Hence we have to prove

$$\left|1 - 2\sum_{i=1}^{n}(x_{2i})(x_{2i+1} - x_{2i-1})\right| = \left|\sum_{i=1}^{n}(x_{2i+1} + x_{2i-1} - 2x_{2i})(x_{2i+1} - x_{2i-1})\right| \le h$$

Now

$$\left|\sum_{i=1}^{n}(x_{2i+1} + x_{2i-1} - 2x_{2i})(x_{2i+1} - x_{2i-1})\right| \le \sum_{i=1}^{n}\left|x_{2i+1} + x_{2i-1} - 2x_{2i}\right|(x_{2i+1} - x_{2i-1})$$

$$\le \sum_{i=1}^{n} h(x_{2i+1} - x_{2i-1}) = h$$

because $|x_{2i+1} + x_{2i-1} - 2x_{2i}| \le \max(x_{2i+1} - x_{2i}, x_{2i} - x_{2i-1})$. Equality can not occur, because we would need $x_{2i+1} - x_{2i}$ or $x_{2i} - x_{2i-1}$ would have to be zero in that case. $\square$

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Problem 2 *Suppose that* (a_n) *is a sequence of real numbers such that the series*

$$\sum_{n=1}^{\infty} \frac{a_n}{n}$$

is convergent. Show that the sequence

$$b_n = \frac{\sum\limits_{j=1}^{n} a_j}{n}$$

is convergent and find its limit.

Solution (by Stijn Cambie) Write $A_n = \sum_{i=1}^{n} \frac{a_i}{i}$. Suppose this converges to A. We have $b_n = A_n - \frac{\sum_{i=1}^{n-1} A_i}{n}$. This converges to zero as $n \longrightarrow \infty$. Indeed, for each $\epsilon > 0$ take some I_0 such that $|A_i - A| \leq \frac{\epsilon}{3}$ for $i \geq I_0$ and take $n_0 > I_0$ such that

$$(n_0 - I_0 + 1)\frac{\epsilon}{3} + \sum_{i=1}^{I_0 - 1} |A - A_i| < n_0 \frac{\epsilon}{2}$$

and $|\frac{A}{n_0}| < \frac{\epsilon}{6}$.

Then for $n > n_0$ we have

$$|b_n| = \left| A_n - \frac{\sum_{i=1}^{n-1} A_i}{n} \right| \leq |A_n - A| + \left|\frac{A}{n}\right| + \frac{\sum_{i=1}^{n-1} |A - A_i|}{n} < \frac{\epsilon}{3} + \frac{\epsilon}{2} + \frac{\epsilon}{6}$$

□

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Problem 3 *Two players play the following game: Let n be a fixed integer greater than 1. Starting from number $k = 2$, each player has two possible moves: either replace the number k by $k + 1$ or by $2k$. The player who is forced to write a number greater than n loses the game. Which player has a winning strategy for which n?*

Solution (by Stijn Cambie)

Write n in base 4. We will prove that the second player B can only win when the representation contains only 0 and 2s. The first player A wins in the other cases.

Claim 1 *Person A wins for n odd.*

Proof He just has to do $k \to k + 1$ in each step, in each move he makes an even number odd. Next B can make the number only even. As n is odd, A won't ever make an even number smaller than n bigger than n by adding one. Hence B has to do this and will lose. □

B wins for $n = 2$, trivial. When A chooses $2 \to 4$ in his first step, he wins for $n = 4, 6$. Person B can win for 8 by multiply the number of A by 2 in his first step and then both have to add one each step and B reaches 8. Hence the base cases are correct.

Claim 2 *When person X wins for n, he can also win for $4n$ and $4n + 2$.*

Proof At some step, the other person will get a number k bigger than n, next X makes $2k > 2n + 1$ and by alternating adding one, we see X will reach every number. □

Hence if A wins for some n, he wins also for $4n, 4n + 1, 4n + 2, 4n + 3$ by both claims. So player B can only win for $4n, 4n + 2$ where n is a number that has the predicted representation (and so do $4n, 4n + 2$). □

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Problem 4 *Let $A = [a_{ij}]_{n\times n}$ be a matrix with nonnegative entries such that*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = n\,.$$

1. *Prove that $|\det A| \le 1$.*
2. *If $|\det A| = 1$ and $\lambda \in \mathbb{C}$ is an arbitrary eigenvalue of A, show that $|\lambda| = 1$.*

(We call $\lambda \in \mathbb{C}$ an eigenvalue of A if there exists a non-zero vector $x \in \mathbb{C}^n$ such that $Ax = \lambda x$.)

Solution (by Stijn Cambie)

1. We prove that the statement is true and equality occur only for a permutation matrix.
   We can prove this by induction.
   For $n = 1$ this is trivial (we have $\det A = 1$).
   For $n = 2$ we have for $A = \begin{Bmatrix} a & b \\ c & d \end{Bmatrix}$ that $|\det(A)| = |ad - bc| \le (\frac{a+d}{2})^2 + (\frac{b+c}{2})^2 \le (\frac{a+d+b+c}{2})^2 = 1$.
   Equality occurs only when $a = d = 1$ or $b = c = 1$. So the induction hypothesis is proven for $n \le 2$.

   Induction step:

   Assume the sum of the entries in the $(n+1)$-th row of A is x. The sum of all other entries is $n + 1 - x$.
   By homogenizing and using the induction hypothesis, we have that for each minor the determinant of it has absolute value $\le (\frac{|n+1-x|}{n})^n$.
   Now $|\det A| \le \sum |a_{(n+1)j}||\det M_{(n+1)j}| \le \frac{nx}{n}(\frac{|n+1-x|}{n})^n \le \left(\frac{nx+n(n+1-x)}{n(n+1)}\right)^{n+1} = 1$ by $\mathrm{AM} - \mathrm{GM}$.
   Equality could only occur when $x = 1$ and each $|\det M_{(n+1)j}| = 1$ when $|a_{(n+1)j}| > 0$ so each minor has to be a permutation matrix, which is possible only once.
   Hence A is also a permutation matrix.

2. If λ is an eigenvalue and an eigenvector is $(x_1 x_2 \cdots x_n)^T$, then $\lambda x_i = x_j$ for some i, j, such that $x_i \ne 0$.
   Repeating this, we get cycles such that $x_i = \lambda^m x_i$ for some m and hence $\lambda^m = 1$, hence $|\lambda| = 1$.

□

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Problem 1

1. *Let u and v be two nilpotent elements in a commutative ring (with or without unity). Prove that $u+v$ is also nilpotent. (An element u is called nilpotent if there exists a positive integer n for which $u^n = 0$.)*
2. *Show an example of a (non-commutative) ring R and nilpotent elements $u, v \in R$ such that $u + v$ is not nilpotent.*

Solution (by Stijn Cambie)

1. As u, v are nilpotent, there exist n, m such that $u^n = 0 = v^m$. This means $u^t = 0$ for all $t \geq n$ and $v^s = 0$ for all $s \geq m$. Next $u+v$ is nilpotent as $(u+v)^{n+m} = \sum \binom{n+m}{i} u^i v^{n+m-i} = 0$,
because each term $u^i v^{n+m-i} = 0$ as $i \geq n$ or $n+m-i \geq m$ so u^i or $v^{n+m-i} = 0$ in each summand.
2. Take the ring $R = (\mathbb{Z}^{2*2}, +, \cdot)$ with elements $u = \begin{Bmatrix} 1 & -1 \\ 1 & -1 \end{Bmatrix}$ and $v = \begin{Bmatrix} -1 & -1 \\ 1 & 1 \end{Bmatrix}$.

   Then $u^2 = v^2 = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \end{Bmatrix}$ while $u + v = \begin{Bmatrix} 0 & -2 \\ 2 & 0 \end{Bmatrix}$ is not nilpotent as $(u+v)^n = \begin{Bmatrix} 0 & (-2)^n \\ 2^n & 0 \end{Bmatrix}$.

□

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Problem 2 *Let $(G, \cdot)$ be a finite group of order n. Show that each element of G is a square if and only if n is odd.*

Solution (by Stijn Cambie)

1. If n is odd, we know by Lagrange's theorem that for every element $g \in G: |g| = \text{ord}(g)$ divides $|G| = n$ and hence $|g|$ is also odd. Write $t = \frac{|g|+1}{2}$, then g is the square of g^t. As g was taken arbitrary, it holds for every element of G.

2. If n is even, we have to find at least one element which isn't a square.
   Claim There exist some element with order 2.

Proof Suppose the contrary. We know that the inverse in a group is unique and 1 is its own inverse. For every other element g, we would have $g \neq g^{-1}$ as $g^2 = 1$ means $|g| = 1, 2$ and $|g| = 1$ is only possible for 1. Now look at the sets $\{g, g^{-1}\}$. A $\{1, 1\}$ contains only one element and every other set contains 2 elements, we would have split up G in one one-element-set and two-element-sets, which is impossible as $2 \mid \text{ord}(G)$.

Hence there is at least yet one element g such that $g = g^{-1}$ and hence $g^2 = 1$. $\square$

Because G is finite, we can write all orders of the different elements. Take the maximum $m > 0$ of $\{v_2(|g|) \mid g \in G\}$. Next, choose an element $h \in S$ such that $2^m \mid \text{ord}(h) = 2t$. Suppose h is a square, we have $h = k^2$ for some $k \in G$. Then we have $h^{2t} = k^{4t} = 1$, so $\text{ord}(k) \mid 4t$. Next $k^{2t} = h^t \neq 1$ as the order of h is $2t$. This means $\text{ord}(k) \mid 4t$ and $\text{ord}(k) \nmid 2t$ hence $v_2(k) = v_2(4t) = m + 1$. This is in contradiction with the way we have chosen m. Hence h is an element of G which is not a square. So if n is even, not all elements are squares.

$\square$

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Problem 3 *For a function* $f\colon [0,1] \to \mathbb{R}$ *the secant of* f *at points* $a, b \in [0,1]$, $a < b$, *is the line in* $\mathbb{R}^2$ *passing through* $\big(a, f(a)\big)$ *and* $\big(b, f(b)\big)$. *A function is said to intersect its secant at* a, b *if there exists a point* $c \in (a, b)$ *such that* $\big(c, f(c)\big)$ *lies on the secant of* f *at* a, b.

1. *Find the set* $\mathcal{F}$ *of all continuous functions* f *such that for any* $a, b \in [0,1], a < b$, *the function* f *intersects its secant at* a, b.
2. *Does there exist a continuous function* $f \notin \mathcal{F}$ *such that for any rational* $a, b \in [0,1]$, $a < b$, *the function* f *intersects its secant at* a, b?

Solution

$\square$

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Problem 4 *Let* $f\colon [0,+\infty) \to \mathbb{R}$ *be a strictly convex continuous function such that*

$$\lim_{x\to+\infty} \frac{f(x)}{x} = +\infty\,.$$

Prove that the improper integral $\int_0^{+\infty} \sin\bigl(f(x)\bigr)\,\mathrm{d}x$ *is convergent but not absolutely convergent.*

Solution

□