The 15th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 6th April 2005
Category II

Problem 1. For an arbitrary square matrix M, define

$$\exp(M) = I + \frac{M}{1!} + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots .$$

Construct 2×2 matrices A and B such that $\exp(A+B) \neq \exp(A)\exp(B)$.

[10 points]

Problem 2. Let $(a_{i,j})_{i,j=1}^{n}$ be a real matrix such that $a_{i,i} = 0$ for $i = 1, 2, \ldots, n$. Prove that there exists a set $\mathcal{J} \subset \{1, 2, \ldots, n\}$ of indices such that

$$\sum_{\substack{i \in \mathcal{J} \\ j \notin \mathcal{J}}} a_{i,j} + \sum_{\substack{i \notin \mathcal{J} \\ j \in \mathcal{J}}} a_{i,j} \geq \frac{1}{2} \sum_{i,j=1}^{n} a_{i,j} .$$

[10 points]

Problem 3. Let $f\colon [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Find the limit

$$\lim_{n\to\infty} \left(\frac{(2n+1)!}{(n!)^2} \right)^2 \int_0^1 \int_0^1 \big(xy(1-y)(1-y)\big)^n f(x,y)\, \mathrm{d}x\, \mathrm{d}y .$$

[10 points]

Problem 4. Let R ba a finite ring with the following property: for any $a, b \in R$ there exists an element $c \in R$ (depending on a and b) such that $a^2 + b^2 = c^2$. Prove that for any $a, b, c \in R$ there exists $d \in R$ such that $2abc = d^2$.

(Here $2abc$ denotes $abc + abc$. The ring R is assumed to be associative, but not necessarily commutative and not necessarily containing a unit.) [10 points]