The 15th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 6th April 2005
Category I

**Problem 1.** Let $S_0 = \{z \in \mathbb{C} : |z| = 1, z \neq -1\}$ and $f(z) = \operatorname{Im} z/(1 + \operatorname{Re} z)$. Prove that $f$ is a bijection between $S_0$ and $\mathbb{R}$. Find $f^{-1}$. [10 points]

**Problem 2.** Let $f \colon A^3 \to A$ where $A$ is a nonempty set and $f$ satisfies:

(a) for all $x, y \in A$, $f(x, y, y) = x = f(y, y, x)$ and

(b) for all $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \in A$,

$$f\big(f(x_1, x_2, x_3), f(y_1, y_2, y_3), f(z_1, z_2, z_3)\big) = \\ = f\big(f(x_1, y_1, z_1), f(x_2, y_2, z_2), f(x_3, y_3, z_3)\big)\,.$$

Prove that for an arbitrary fixed $a \in A$, the operation $x + y = f(x, a, y)$ is an Abelian group addition. [10 points]

**Problem 3.** Find all reals $\lambda$ for which there is a nonzero polynomial $P$ with real coefficients such that

$$\frac{P(1) + P(3) + P(5) + \cdots + P(2n-1)}{n} = \lambda P(n)$$

for all positive integers $n$, and find all such polynomials for $\lambda = 2$. [10 points]

**Problem 4.** Let $(x_n)_{n\geq 2}$ be a sequence of real numbers such that $x_2 > 0$ and $x_{n+1} = -1 + \sqrt[n]{1 + nx_n}$ for $n \geq 2$. Find

(a) $\lim\limits_{n\to\infty} x_n\,,$

(b) $\lim\limits_{n\to\infty} nx_n\,.$

[10 points]