The 14^{th} Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31^{st} March 2004
Category I

Problem 1 *Suppose that* $f\colon [0,1] \to \mathbb{R}$ *is a continuously differentiable function such that* $f(0) = f(1) = 0$ *and* $f(a) = \sqrt{3}$ *for some* $a \in (0,1)$. *Prove that there exist two tangents to the graph of* f *that form an equilateral triangle with an appropriate segment of the* x*-axis.*

Solution Let $k\colon [0,1] \to \mathbb{R}$ be defined as $k(x) = f'(x)$. We know that k is continuous. Let $b \in (0,1)$ be the point where f reaches its maximum value. Then $k(b) = f'(b) = 0$ and $f(b) \geq \sqrt{3} > 0$ (as $f(a) = \sqrt{3}$). Since $f(0) = 0$ and $f(b) \geq \sqrt{3}$, there exists an $x_0 \in [0,b)$ such that $f'(x_0) = k(x_0) \geq \sqrt{3}/b > \sqrt{3}$. As k is continuous, we can find an $y_1 \in (x_0, b)$ such that $k(y_1) = f'(y_1) = \sqrt{3}$. Using the same argument, there exists an $y_2 \in (b, 1]$ such that $k(y_2) = f'(y_2) = -\sqrt{3}$. It is easy to see that the tangents to the graph of f at y_1 and y_2 with the appropriate segment of the x-axis form an equilateral triangle. $\square$

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Problem 2 *Evaluate the sum*

$$\sum_{n=0}^{\infty} \arctan\left(\frac{1}{1+n+n^2}\right).$$

Solution Using the formula for the diference of two arcustangents

$$\arctan u - \arctan v = \arctan \frac{u-v}{1+uv},$$

(this formula is valid whenever $uv \neq -1$) we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \arctan\left(\frac{1}{1+n+n^2}\right) &= \arctan(1) + \sum_{n=1}^{\infty} \arctan\left(\frac{(n+1)-n}{1+n(n+1)}\right) = \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty} \arctan\left(\frac{(n+1)-n}{1+n(n+1)}\right) = \\
&= \frac{\pi}{4} + \sum_{n=1}^{\infty} \arctan\left(\frac{1}{n}\right) - \arctan\left(\frac{1}{n+1}\right) = \frac{\pi}{2}.
\end{aligned}$$

□

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Problem 3 *Denote by $B(c,r)$ the open disc in the plane of center c and radius r. Prove or disprove that there exists a sequence $\{z_n\}_{n=1}^{\infty}$ in $\mathbb{R}^2$ such that the open discs $B(z_n, 1/n)$ are pairwise disjoint and the sequence $\{z_n\}_{n=1}^{\infty}$ is convergent.*

Solution Both statemets are true, it is sufficient to prove (b) (in general version, it is obviously not true in $\mathbb{R}^1$ due to measure argument, and in $\mathbb{R}^k$, if $k \geq 2$, it is sufficient to prove the positive answer for $k = 2$). We will prove the crucial Lemma:

Lemma *Let $a > 0$ and let $C = (0,a)^2$. Then there exists an $n_a \in \mathbb{N}$ and a sequence $\{t_n\}_{n \geq n_a} \subset C$ such that the open balls $B(t_n, 1/n) \subset C$ for $n \geq n_a$ are mutually disjoint.*

Proof Consider the numbers $n \in \mathbb{N}$ and $c > 0$ which will be specified concretely later and suppose that $n > 4$ and $n(c-1) > 1$. Let us denote $n_k = \lfloor nc^k \rfloor$. We have $n_0 = n$ and as $n(c-1) > 1$, we also have that the sequence $\{n_k\}$ is strictly increasing. For every $k \geq 0$, we denote

$$s_k = \frac{2}{n_k} + \cdots + \frac{2}{n_{k+1} - 1}\,.$$

By easy integral estimate, we have

$$s_k \leq 2\bigl(\ln(n_{k+1} - 1) - \ln(n_k - 1)\bigr) \leq 2 \ln \frac{nc^{k+1}}{nc^k - 2} = 2 \ln \frac{c}{1 - 2/n} =: d(c,n)\,.$$

Now we easilly see that for every $k \geq 0$, we can find mutually disjoint open balls of diameters $1/n_k, \ldots, 1/(n_{k+1} - 1)$ inside the rectangle of width $d(c,n)$ and height $2/n_k$. Since

$$\sum_{k=0}^{\infty} \frac{2}{n_k} = 2 \sum_{k=0}^{\infty} \frac{1}{\lfloor nc^k \rfloor} \leq 4 \sum_{k=0}^{\infty} \frac{1}{nc^k} = \frac{4c}{n(c-1)}\,,$$

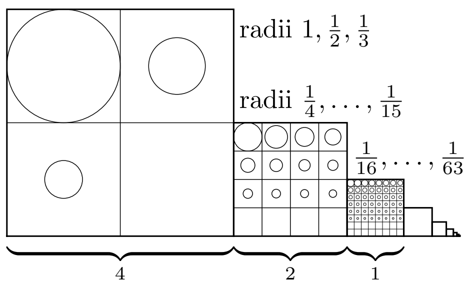
it is clear that we can find mutually disjoint open balls of diameters $1/n, 1/(n+1), \ldots$ inside the rectangle of width $d(c,n)$ and height $4c/\bigl(n(c-1)\bigr)$.[1] For the given a, now it is sufficient to set $c > 1$ so that $2 \ln c < a^2$ and consequently set $n_a = n$ so that $n > 4$, $n(c-1) > 1$ and

$$\max\left\{ d(c,n), \frac{4c}{n(c-1)} \right\} < a\,.$$

The lemma is proved. □

Now by means this Lemma, we can construct the sequence z_n easilly. Let $a_i = 2^{-i}$ and let $C_i = (a_i, 2a_i)^2$. By the lemma, we can find an increasing sequence of the numbers $n_i = n_{a_i}$ such that we can place open balls of the diameters $1/n_0, 1/(n_0+1), \ldots, 1/(n_1 - 1)$ inside C_0 in such a way that they are mutually disjoint. Then we place open balls of the diameters $1/n_1, \ldots, 1/(n_2 - 1)$ inside C_1 so that they are mutually disjoint, etc. The first $n_0 - 1$ balls can be placed anywhere else. It is clear that the sequence of the centres of thse balls is convergent (having the limit 0). □

Solution Take squares with lenghts of sides $4, 2, 1, \frac{1}{2}, \ldots$ and make a row of them. Put discs with radii $1, \frac{1}{2}, \frac{1}{3}$ to first square. It is possible since it can be divided into 4 smaller squares with side 1. Than put disc with radii $\frac{1}{4}, \ldots, \frac{1}{15}$ to second square. Thanks to dividing to 16 smaller squares it is again possible. Continuing by this way we put all discs into sequence of convergent squares, therefore also sequence of discs is convergent.

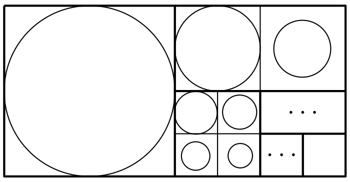


[1]Recall that $n = n_0$. Place the balls of diameters $1/n_0, \ldots, 1/(n_1 - 1)$ into a rectangle of width $d(c,n)$ and height $2/n_0$. Place the balls of diameters $1/n_1, \ldots, 1/(n_2 - 1)$ into a rectangle of width $d(c,n)$ and height $2/n_1$. Etc.

[2]In order that it is possible to ensure $d(c,n) < a$ by the choice of large n.

□

Solution Take rectangle 4×2 and divide it into rectangles 2×2, 2×1, 1×1, $1 \times \frac{1}{2}$, ... To first rectangle put disc with radius 1, to second rectangle discs with radii $\frac{1}{2}$ and $\frac{1}{3}$, to third one discs with radii $\frac{1}{4}, \dots, \frac{1}{7}$, and so on. Sequence of rectangles is again convergent and therefore also sequence of discs.



□

Solution Let $z_n = (x_n, y_n)$, where $x_n = \sum_{i=1}^{\lfloor\sqrt{n}\rfloor} \frac{6}{i^2}$ and $y_n = \sum_{i=\lfloor\sqrt{n}\rfloor^2}^{n} \frac{3}{i}$. From real analysis we know $\lim_{n\to\infty} x_n = \pi^2$ and by small counting $\lim_{n\to\infty} y_n = 0$, so $\{z_n\}_{n=1}^{\infty}$ converges to $(\pi^2, 0)$.

Let's prove that $B_n(z_n, \frac{1}{n})$ and $B_m(z_m, \frac{1}{m})$ are disjont for all $m \neq n$. The discs are grouped in columns. In each column are discs from B_{n^2} to B_{n^2+2n}, so each column of discs is included in one of zones

$$A_n = \left\{(x, y) : x \in \left(\left(\sum_{i=1}^{n} \frac{6}{i^2}\right) - \frac{1}{n^2}, \left(\sum_{i=1}^{n} \frac{6}{i^2}\right) + \frac{1}{n^2}\right)\right\}.$$

These A_n are pairwise disjont, because

$$\left(\sum_{i=1}^{n} \frac{6}{i^2}\right) + \frac{1}{n^2} < \left(\sum_{i=1}^{n+1} \frac{6}{i^2}\right) - \frac{1}{(n+1)^2}$$

so all of the columns are pairwise disjoint. Discs in each column are pairwise disjoint (if B_n and B_{n+1} are in the same column, then $|y_{n+1} - y_n| \geq \frac{3}{n+1} \geq \frac{1}{n} + \frac{1}{n+1}$). Result follows: there exists desired sequence. □

Problem 4 *Find all pairs (m,n) of positive integers such that $m+n$ and $mn+1$ are both powers of 2.*

Solution The set of such pairs is

$$L = \{(2^t-1, 2^t+1),\ (2^t+1, 2^t-1),\ (2^t-1, 1),\ (1, 2^t-1)\}$$

where t runs through the set of all positive integers.

For the proof, let m, n, k, l be integers such that $m, n \geq 1$ and

$$mn + 1 = 2^l\,,$$
$$m + n = 2^k\,.$$

From $m, n \geq 1$ we get $2^l = mn + 1 \geq 2$, so $l \geq 1$ and 2^l is even. Hence m and n are both odd. Assume $m = n$. Then $m = n = 2^{k-1}$ must be odd, hence $k = 1$, which implies $m = n = 1$, and $(1,1) \in L$. From now on, we assume $m < n$, so $y = \frac{1}{2}(n-m)$ is a positive integer.

If $y = 1$, then we get $n - m = 2$, $m + n = 2^k$, so $n = 2^{k-1} + 1$ and $m = 2^{k-1} - 1$, which yields a pair in the set L (if $k > 1$ as m is not positive otherwise).

Let $y > 1$. Then

$$0 < y^2 - 1 = \left(\frac{n+m}{2}\right)^2 - mn - 1 = 2^{2k-2} - 2^l\,.$$

This implies $2k - 2 > l$, and hence $y^2 - 1 = (y-1)(y+1)$ is divisible by 2^l. The greatest common divisor of $y-1$ and $y+1$ is 2, so one of these positive factors must be divisible by 2^{l-1}. In both cases, we get $y+1 \geq 2^{l-1}$. Hence

$$2^{2k-2} - 2^l = (y+1)(y-1) \geq 2^{l-1}(2^{l-1} - 2) = 2^{2l-2} - 2^l\,,$$

so $2k - 2 \geq 2l - 2$, which implies $k \geq l$. But then

$$0 \leq (m-1)(n-1) = (mn+1) - (m+n) = 2^l - 2^k \leq 0\,,$$

so we must have equality. Thus $l = k$ and (since $m < n$) $m = 1$. From this we get $n = 2^k - 1$, so $(m,n) \in L$. $\square$

Solution Let us write $m+n = 2^a$ and $mn+1 = 2^b$. Of course $a, b \geq 1$ and $b \geq a$ since $2^b - 2^a = (m-1)(n-1) \geq 0$. If $b = a$, then $(m-1)(n-1) = 0$, so

$$m = 1 \text{ and } n = 2^a - 1 \qquad \text{or} \qquad m = 2^a - 1 \text{ and } n = 1.$$

So, suppose $b > a$. If $a = 1$, then $m = n = 1$. If $a = 2$, then $\{m,n\} = \{1,3\}$. So, let also $a \geq 3$. It is obvious, that m and n are odd (since $mn+1$ is even). If $m \equiv n \equiv 3 \pmod 4$, then $0 \equiv 2^a \equiv m + n \equiv 2 \pmod 4$, contradiction. So thanks to symetry we can assume that $m \equiv 1 \pmod 4$. Since $m + n \equiv 0 \pmod 4$ we must have $n \equiv 3 \pmod 4$. So, let $m = 4c+1$ and $n = 4d+3$, where $c, d \in \{0, 1, \dots\}$. We have

$$(m+1)(n+1) = 2^a + 2^b = 2^a(2^{b-a} + 1)$$

thus

$$8(2c+1)(d+1) = 2^a(2^{b-a}+1)\,.$$

Since $2c+1$ is odd, we have $2^{a-3} \mid d+1$. So let $d + 1 = x \cdot 2^{a-3}$, with $x \in \{1, 2, \dots\}$. Thus, $n = 4(d+1) - 1 = x2^{a-1} - 1$ and so $m + x2^{a-1} - 1 = 2^a$. Since $m \geq 1$, we have $x \leq \frac{2^a}{2^{a-1}} = 1$ and so $x = 1$ or $x = 2$. If $x = 1$, then $m = 2^{a-1} + 1$ and $n = 2^{a-1} - 1$. If $x = 2$, then $m = 1$ and $n = 2^a - 1$. So, we have proved that it is

$$\text{either} \quad \{m,n\} = \{1, 2^a - 1\} \quad \text{or} \quad \{m,n\} = \{2^a + 1, 2^a - 1\}\,, \quad \text{where } a \in \mathbb{N}.$$

$\square$

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Category II

Problem 1 *Are the groups* $(\mathbb{Q},+)$ *and* $(\mathbb{Q}^+,\cdot)$ *isomorphic? (The symbol* $\mathbb{Q}^+$ *denotes the set of all positive rational numbers.)*

Solution Assume they are, and name the isomorphism φ. Then there exists an $a \in \mathbb{Q}$ such that $\varphi(a) = 2$. Let $\varphi\left(\frac{a}{2}\right) = b$. Then

$$2 = \varphi(a) = \varphi\left(\frac{a}{2} + \frac{a}{2}\right) = \varphi\left(\frac{a}{2}\right) \cdot \varphi\left(\frac{a}{2}\right).$$

Therefore, $b = \varphi\left(\frac{a}{2}\right) = \sqrt{2}$, a contradiction. □

Solution Suppose (for contradiction) that there exists an isomorfism $f\colon (\mathbb{Q},+) \to (\mathbb{Q}^+,\cdot)$. Then $f(0) = 1$, because 0 and 1 are corresponding neutral elements of groups $(\mathbb{Q},+)$ and $(\mathbb{Q}^+,\cdot)$. Let $f(1) = \frac{a}{b}$, $a, b \in \mathbb{N}$, $(a,b) = 1$. Similarly, let for every $n \in \mathbb{N}$, $n \geq 2$, $f\left(\frac{1}{n}\right) = \frac{a_n}{b_n}$, where $a_n, b_n \in \mathbb{N}$, $(a_n, b_n) = 1$. Then it is

$$f(1) = f\left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right) = \left(f\left(\frac{1}{n}\right)\right)^n = \left(\frac{a_n}{b_n}\right)^n = \frac{a}{b}.$$

It is still $\big((a_n)^n, (b_n)^n\big) = 1$, so consequently $a = (a_n)^n$ and $b = (b_n)^n$. If there is $m \in \mathbb{N}$ such that $a_m \geq 2$, then $a \geq 2$ and $a_n \geq 2$ for all n. That follows $a \geq 2^n$ for all n, contradiction. Thus for every $n \in \mathbb{N}$ it holds $a_n = 1$, specially $a = 1$. Same argument gives $b = 1$. But then $f(1) = \frac{a}{b} = 1 = f(0)$, f is not injective, contradiction. □

Solution Suppose there exists an isomorphism $\sigma\colon (\mathbb{Q},+) \to (\mathbb{Q}^+,\cdot)$. Consider homomorphism $\tau\colon (\mathbb{Q},+) \to (\mathbb{Q},+)$ given by equation $\tau(q) = 2q$ for $q \in \mathbb{Q}$, and corresponding homomorphism $\tau'\colon (\mathbb{Q}^+,\cdot) \to (\mathbb{Q}^+,\cdot)$ given by equation $\tau'(q) = q^2$ for $q \in \mathbb{Q}^+$. It follows that

$$\begin{aligned} \sigma \circ \tau(q) &= \sigma(2q) = \big(\sigma(q)\big)^2 \\ \tau' \circ \sigma(q) &= \tau'\big(\sigma(q)\big) = \big(\sigma(q)\big)^2 \end{aligned}$$

for all $q \in \mathbb{Q}$. It is clear that homomorphism τ is isomorphism, so also $\tau \circ \sigma = \tau' \circ \sigma$ is isomorphism, but this contradicts the fact, that τ' is not injective (that is obvious) and therefore $\tau' \circ \sigma$ cannot be isomorphism. □

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Problem 2 *Find all functions* $f\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *such that*

1. $f(x,0) = f(0,x) = x$ *for all* $x \in \mathbb{R}_0^+$,
2. $f(f(x,y),z) = f(x,f(y,z))$ *for all* $x,y,z \in \mathbb{R}_0^+$ *and*
3. *there exists a real* k *such that* $f(x+y,x+z) = kx + f(y,z)$ *for all* $x,y,z \in \mathbb{R}_0^+$.

(The symbol $\mathbb{R}_0^+$ *denotes the set of all non-negative real numbers.)*

Solution Take $x \le y$. Then

$$\begin{aligned} f(x,y) &= f(x+0, x+(y-x)) = kx + f(0,y-x) = kx + (y-x) = (k-1)x + y = \\ &= f(x+(y-x), x+0) = f(y,x)\,. \end{aligned} \tag{1}$$

Let $x \le y$. If $z \ge y + (k-1)x$, then

$$f(f(x,y),z) = f(y+(k-1)x, z) = (k-1)(y+(k-1)x) + z\,.$$

Further, if $z \ge y$ and $z + (k-1)y \ge x$, then

$$f(x, f(y,z)) = f(x, (k-1)y + z) = (k-1)x + (k-1)y + z\,.$$

If we take z so large that all the above inequalities are satisfied, then we have

$$(k-1)(y+(k-1)x) + z = (k-1)x + (k-1)y + z\,,$$

or $(k-1)^2 x = (k-1)x$, and the only possibilities are $k = 1, 2$. Substituting into (1), we obtain $f(x,y) = x+y$ for $k=2$, and $f(x,y) = \max\{x,y\}$ for $k=1$ (we may use (1) only if $y \ge x$, whence the maximum). It is easy to check, that both functions satisfy all conditions. □

Solution Consider two cases $x \le y$ and $x \ge y$:

$$\begin{aligned} x \le y\colon\quad f(x,y) &= f\big(x, x+(y-x)\big) = kx + f(0,y-x) \\ &= kx + y - x = (k-1)x + y \\ x \ge y\colon\quad f(x,y) &= f\big(y+(x-y), y\big) = ky + f(x-y,0) \\ &= ky + x - y = (k-1)y + x \end{aligned}$$

So it is

$$f(x,y) = (k-1)\min\{x,y\} + \max\{x,y\}.$$

Because $f(x,x) = kx \ge 0$, we have $k \ge 0$. Consider now the second equality from the problem: $f\big(f(x,y),z\big) = f\big(x,f(y,z)\big)$.

$$f\big(f(1,1),k\big) = f(k,k) = k^2$$

$$f\big(1,f(1,k)\big) = \begin{cases} f(1,2k-1) = (k-1)\min\{1,2k-1\} + \max\{1,2k-1\} & \\ \qquad = (k-1) + (2k-1) & \text{for } k \ge 1 \\ f(1,k^2-k+1) = (k-1)\min\{1,k^2-k+1\} + \max\{1,k^2-k+1\} & \\ \qquad = (k-1)(k^2-k+1) + 1 & \text{for } k \le 1 \end{cases}$$

Consider cases $k \ge 1$ and $k \le 1$ separatelly.

$$\begin{aligned} k \ge 1\colon\qquad k^2 &= (k-1) + (2k-1) \\ k^2 - 3k + 2 &= 0 \\ (k-2)(k-1) &= 0 \\ k &= 1,2 \text{ or some of them} \\ k \le 1\colon\qquad k^2 &= (k-1)(k^2-k+1) + 1 \\ k^3 - 3k^2 + 2k &= 0 \\ k(k-2)(k-1) &= 0 \\ k &= 0,1,2 \text{ or some of them} \end{aligned}$$

For $k = 0, 1, 2$ the functions have the following form.

$$\begin{aligned} k = 2: &\quad f(x,y) = (k-1)\min\{x,y\} + \max\{x,y\} = x + y \\ k = 1: &\quad f(x,y) = \max\{x,y\} \\ k = 0: &\quad f(x,y) = |x - y| \end{aligned}$$

First two functions satisfy to all three conditions of the problem (easy to check), the last one does not fulfil the second condition for triple $(x, y, z) = (1, 2, 3)$. $\square$

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Category II

Problem 3 *Let* $\sum\limits_{n=1}^{\infty} a_n$ *be a divergent series with positive nonincreasing terms. Prove that the series*

$$\sum_{n=1}^{\infty} \frac{a_n}{1+na_n}$$

diverges.

Solution Let (d_n) be a sequence defined as follows:

$$1, 1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 0, \dots$$

(a block of 2^n ones is followed by a block of 2^n zeros, and that block of 2^n zeros is followed by a block of 2^{n+1} ones and so forth).

Define sets

$$\begin{aligned}
&D = \{\, n \in \mathbb{N} : d_n = 1 \,\}\,,\\
&C = 2D\,,\\
&E = 2\mathbb{N} \setminus C\,,\\
&B = C \cup (E+1)\,,\\
&A = C \cup B_1 = 2\mathbb{N}
\end{aligned}$$

and define sequences $x_n = \chi_A(n)$ and $y_n = \chi_B(n)$ where χ_S denotes the indicator function of a set S.

Then

$$\lim_{n\to\infty} \frac{x_0 + x_1 + \cdots + x_n}{n+1} = \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{y_0 + y_1 + \cdots + y_n}{n+1} = \frac{1}{2}$$

and the sequence

$$\lim_{n\to\infty} \frac{x_0 y_0 + x_1 y_1 + \cdots + x_n y_n}{n+1}$$

oscillates between $\frac{1}{4}$ and $\frac{3}{8}$. $\square$

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Category II

Problem 4 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Assume that for every $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ (depending on x) such that*

$$f^{(n)}(x) = 0\,.$$

Prove that f is a polynomial.

Solution Denote

$$E_n := \{\, x \in \mathbb{R}\,;\, f^{(n)}(x) = 0 \,\}\,.$$

Note that each of the sets E_n is closed due to continuity of $f^{(n)}$. The assumption implies

$$\mathbb{R} = \bigcup_{n\in\mathbb{N}} E_n\,.$$

By Baire's theorem, there is an interval I such that some E_n is dense in I.[3] Since $f^{(n)}$ is a continuous function, we obtain

$$f^{(n)}(x) = 0 \qquad \text{for all } x \in I.$$

Therefore, the function f is a polynomial on the interval I. Now, we denote $\{I_\lambda\}_{\lambda\in\Lambda}$ the set of all maximal open intervals I_λ such that f is a polynomial on I_λ. (Here, Λ is an index set.) We have already proved at least one such interval I_λ exists. Moreover, all the sets I_λ are clearly mutually disjoint and their union is dense in $\mathbb{R}$ (otherwise, we can repeat the previous argument using Baire's theorem in the interval $J \subseteq \mathbb{R} \setminus \bigcup_{\lambda\in\Lambda} I_\lambda$).

Now, let us consider the set

$$H := \mathbb{R} \setminus \bigcup_{\lambda\in\Lambda} I_\lambda\,.$$

First, we prove that this set has no isolated points. If there were some isolated point $x \in H$, then there would be two intervals $I_1, I_2 \in \{I_\lambda\}_{\lambda\in\Lambda}$, one of them having x as the right end-point and the other one having x as the left end-point. There would also be some $n \in \mathbb{N}$ such that the n-th derivate vanishes on the union $I_1 \cup I_2$. This would mean that the function f is a polynomial on $I_1 \cup \{x\} \cup I_2$ and the intervals I_1 and I_2 are not maximal.

The set H is a closed subset of the complete space $\mathbb{R}$. Therefore, if it is not empty, it is of second category. Using Baire's theorem again,[4] we prove that there exists an interval J containing at least one point from H and an index $n \in \mathbb{N}$ such that

$$f^{(n)}(x) = 0 \qquad \text{for all } x \in J \cap H.$$

Since every $x \in J \cap H$ is an accumulation point of H (and is an accumulation point of $J \cap H$ therefore, and since $f^{(n+1)}(x)$, i.e. the limit $\lim\limits_{h\to 0}\big(f^{(n)}(x+h) - f^{(n)}(x)\big)/h$ does exist at every point x), we can calculate the $(n+1)$-st derivative of f on $J \cap H$ using just the points from the intersection $J \cap H$. We obtain

$$f^{(n+1)}(x) = 0 \qquad \text{for all } x \in J \cap H$$

and, repeating this argument,

$$f^{(m)}(x) = 0 \qquad \text{for all } x \in J \cap H \text{ and } m \ge n.$$

Now, take any interval $I \in \{I_\lambda\}_{\lambda\in\Lambda}$ such that $I \subseteq J$.[5] There is some index $m \in \mathbb{N}$ such that $f^{(m)}(x) = 0$ for $x \in I$. Let us assume $m > n$. There are two cases: either $f^{(m-1)}$ vanishes (is zero) on the interval I or $f^{(m-1)}$ is a linear function on I. Since the end-points of I belong to H, if follows that $f^{(m-1)}$ is null at these end-points. Therefore, $f^{(m-1)}$ vanishes on the interval I. By induction (repeating this argument), we prove that $f^{(n)}$ vanishes on the interval I in fact. This conclusion (i.e., $f^{(n)} \equiv 0$ on I) is true for every interval $I \in \{I_\lambda\}_{\lambda\in\Lambda}$.

Choose an $x \in J \cap H$. As we already know that $J \cap H$ cannot contain an interval, there must exist two intervals $I_{\lambda_1}, I_{\lambda_2} \in \{I_\lambda\}_{\lambda\in\Lambda}$ such that x is the right end-point of I_{λ_1} and the left end-point of I_{λ_2}. Since

[3] As the sets E_n are closed and $\bigcup_n E_n = \mathbb{R}$, one of the sets must have non-empty interior. Equivalently, there must exist an index n and an interval I such that $I \subseteq E_n$.

[4] We just repeat the above argument: As the sets E_n are closed and $\bigcup_n E_n \supseteq H$, one of the sets E_n must have non-empty interior in H. That is, there exists an interval J whose intersection with H is non-empty such that $J \cap H \subseteq E_n$.

[5] If there were no such an interval I, it would mean that the set $J \cap H$ (hence the set H itself) contains a (closed) interval. But we have already inferred $f^{(n+1)}(x) = 0$ for $x \in J \cap H$. Hence, f is a polynomial on that (closed) interval. That interval should have been included in the collection $\{I_\lambda\}_{\lambda\in\Lambda}$ therefore.

$f^{(n)}$ is continuous and is zero on $I_{\lambda_1} \cup I_{\lambda_2}$, it is zero at x as well. Hence, f is a polynomial on the interval $I_{\lambda_1} \cup \{x\} \cup I_{\lambda_2} \in \{I_\lambda\}_{\lambda \in \Lambda}$, which is a contradiction (all intervals from $\{I_\lambda\}_{\lambda \in \Lambda}$ must be mutually disjoint). This contradiction can be inferred if $H \neq \emptyset$. Consequently, the set H is empty and f is a polynomial on the whole real line. $\square$