The 14th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31st March 2004
Category II

Problem 1. Are the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}^+, \cdot)$ isomorphic? (The symbol $\mathbb{Q}^+$ denotes the set of all positive rational numbers.)

[10 points]

Problem 2. Find all functions $f\colon \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

1. $f(x, 0) = f(0, x) = x$ for all $x \in \mathbb{R}_0^+$,
2. $f(f(x, y), z) = f(x, f(y, z))$ for all $x, y, z \in \mathbb{R}_0^+$ and
3. there exists a real k such that $f(x + y, x + z) = kx + f(y, z)$ for all $x, y, z \in \mathbb{R}_0^+$.

(The symbol $\mathbb{R}_0^+$ denotes the set of all non-negative real numbers.)

[10 points]

Problem 3. Let $\sum_{n=1}^{\infty} a_n$ be a divergent series with positive non-increasing terms. Prove that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + n a_n}$$

diverges. [10 points]

Problem 4. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function. Assume that for every $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ (depending on x) such that

$$f^{(n)}(x) = 0\,.$$

Prove that f is a polynomial. [10 points]