The 14th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 31st March 2004
Category I

Problem 1. Suppose that $f\colon[0,1]\to\mathbb{R}$ is a continuously differentiable function such that $f(0)=f(1)=0$ and $f(a)=\sqrt{3}$ for some $a\in(0,1)$. Prove that there exist two tangents to the graph of f that form an equilateral triangle with an appropriate segment of the x-axis. [10 points]

Problem 2. Evaluate the sum

$$\sum_{n=0}^{\infty}\operatorname{arctg}\left(\frac{1}{1+n+n^2}\right).$$

[10 points]

Problem 3. Denote by $B(c,r)$ the open disk of center c and radius r in the plane. Decide whether there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of points in $\mathbb{R}^2$ such that the open disks $B(z_n,1/n)$ are pairwise disjoint and the sequence $\{z_n\}_{n=1}^{\infty}$ is convergent. [10 points]

Problem 4. Find all pairs (m,n) of positive integers such that $m+n$ and $mn+1$ are both powers of 2. [10 points]