The 13th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2003
Category II

Problem 1. Two real square matrices A and B satisfy the conditions $A^{2002} = B^{2003} = I$ and $AB = BA$. Prove that $A+B+I$ is invertible. (The symbol I denotes the identity matrix.) [10 points]

Problem 2. Let $\{D_1, D_2, \ldots, D_n\}$ be a set of disks in the Euclidean plane. (A disk is a set of points whose distance from the given centre is less than or equal to the given radius.) Let $a_{ij} = S(D_i \cap D_j)$ be the area of $D_i \cap D_j$. Prove that the inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j \geq 0$$

holds for any real numbers $x_1, x_2, \ldots, x_n$. [10 points]

Problem 3. Let $\{a_n\}_{n=0}^{\infty}$ be the sequence of real numbers satisfying $a_0 = 0, \;\; a_1 = 1$ and

$$a_{n+2} = a_{n+1} + \frac{a_n}{2^n}$$

for every $n \geq 0$. Prove that

$$\lim_{n\to\infty} a_n = 1 + \sum_{n=1}^{\infty} \frac{1}{2^{n(n-1)/2} \prod_{k=1}^{n} (2^k - 1)} .$$

[10 points]

Problem 4. Let $f, g\colon [0,1] \to (0, +\infty)$ be two continuous functions such that f and $\frac{g}{f}$ are increasing. Prove that

$$\int_0^1 \frac{\int_0^x f(t)\,\mathrm{d}t}{\int_0^x g(t)\,\mathrm{d}t}\,\mathrm{d}x \leq 2 \int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t\,.$$

[10 points]