The 13th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 2nd April 2003
Category I

Problem 1. Let $d(k)$ denote the number of all natural divisors of a natural number k. Prove that for any natural number n_0 the sequence $\left\{d(n^2+1)\right\}_{n=n_0}^{\infty}$ is not strictly monotone. [10 points]

Problem 2. Let $A = (a_{ij})$ be an $m \times n$ real matrix with at least one non-zero element. For each $i \in \{1, \ldots, m\}$, let $R_i = \sum_{j=1}^{n} a_{ij}$ be the sum of the i-th row of the matrix A, and for each $j \in \{1, \ldots, n\}$, let $C_j = \sum_{i=1}^{m} a_{ij}$ be the sum of the j-th column of the matrix A. Prove that there exist indices $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$ such that

$$a_{kl} > 0\,, \qquad R_k \geq 0\,, \qquad C_l \geq 0\,,$$

or

$$a_{kl} < 0\,, \qquad R_k \leq 0\,, \qquad C_l \leq 0\,.$$

[10 points]

Problem 3. Find the limit

$$\lim_{n\to\infty} \sqrt{1+2\sqrt{1+3\sqrt{\cdots+(n-1)\sqrt{1+n}}}}\,.$$

[10 points]

Problem 4. Let A and B be complex Hermitian 2×2 matrices having the pairs of eigenvalues (α_1, α_2) and (β_1, β_2), respectively. Determine all possible pairs of eigenvalues (γ_1, γ_2) of the matrix $C = A+B$. (We recall that a matrix $A = (a_{ij})$ is Hermitian if and only if $a_{ij} = \overline{a_{ji}}$ for all i and j.) [10 points]