The 11th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 4th April 2001
Category II

Problem 1. Let $n \geq 2$ be an integer and let $x_1, x_2, \ldots, x_n$ be real numbers. Consider $N = \binom{n}{2}$ sums $x_i + x_j,\ 1 \leq i < j \leq n$, and denote them by $y_1, y_2, \ldots, y_N$ (in an arbitrary order). For which n are the numbers $x_1, x_2, \ldots, x_n$ uniquely determined by the numbers $y_1, y_2, \ldots, y_N$? [10 points]

Problem 2. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Define a sequence of functions $f_n\colon [0,1] \to \mathbb{R}$ in the following way:

$$f_0(x) = f(x)\,, \qquad f_{n+1}(x) = \int_0^x f_n(t)\,\mathrm{d}t\,, \qquad n = 0, 1, 2, \ldots\,.$$

Prove that if $f_n(1) = 0$ for all n, then $f(x) \equiv 0$. [10 points]

Problem 3. Let $f\colon (0, +\infty) \to (0, +\infty)$ be a decreasing function which satisfies $\int_0^\infty f(x)\,\mathrm{d}x < +\infty$. Prove that $\lim_{x\to+\infty} x f(x) = 0$. [10 points]

Problem 4. Let R be an associative non-commutative ring and let $n > 2$ be a fixed natural number. Assume that $x^n = x$ for all $x \in R$. Prove that $xy^{n-1} = y^{n-1}x$ holds for all $x, y \in R$. [10 points]