The 10th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 5th April 2000
Category I

Problem 1. Is there a countable set Y and an uncountable family $\mathcal{F}$ of its subsets such that for every two distinct $A, B \in \mathcal{F}$, their intersection $A \cap B$ is finite? [10 points]

Problem 2. Let $f\colon \mathbb{N} \to \mathbb{R}$ be given by

$$f(n) = n^{\frac{1}{2}\tau(n)}$$

for $n \in \mathbb{N} = \{1, 2, \ldots\}$ where $\tau(n)$ is the number of divisors of n. Show that f is an injection into $\mathbb{N}$. [10 points]

Problem 3. Let $a_1, a_2, \ldots$ be a bounded sequence of reals. Is it true that the fact

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = b \qquad \text{and} \qquad \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{a_n}{n} = c$$

implies $b = c$? [10 points]

Problem 4. Let us choose arbitrarily n vertices of a regular $2n$-gon and colour them red. The remaining vertices are coloured blue. We arrange all red-red distances into a non-decreasing sequence and do the same with the blue-blue distances. Prove that the sequences are equal. [10 points]