The 9th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 24th March 1999
Category II

Problem 1. Find the minimal k such that every set of k different lines in $\mathbb{R}^3$ contains either 3 mutually parallel lines or 3 mutually intersecting lines or 3 mutually skew lines. [12 points]

Problem 2. Let $a, b \in \mathbb{R}$, $a \le b$. Assume that $f\colon [a,b] \to [a,b]$ satisfies $\bigl|f(x) - f(y)\bigr| \le |x - y|$ for every $x, y \in [a,b]$. Choose an $x_1 \in [a,b]$ and define

$$x_{n+1} = \frac{x_n + f(x_n)}{2}, \qquad n = 1, 2, 3, \dots .$$

Show that $\{x_n\}_{n=1}^{\infty}$ converges to some fixed point of f. [7 points]

Problem 3. Suppose that we have a countable set A of balls and a unit cube in $\mathbb{R}^3$. Assume that for every finite subset B of A it is possible to put all balls of B into the cube in such a way that they have disjoint interiors. Show that it is possible to arrange all the balls in the cube so that all of them have pairwise disjoint interiors.
[11 points]

Problem 4. Let $u_1, u_2, \dots, u_n \in C([0,1]^n)$ be nonnegative and continuous functions, and let u_j do not depend on the j-th variable for $j = 1, \dots, n$. Show that

$$\left(\int_{[0,1]^n} \prod_{j=1}^{n} u_j \right)^{n-1} \le \prod_{j=1}^{n} \int_{[0,1]^n} u_j^{n-1}.$$

[10 points]