The 9th Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 24th March 1999
Category I

**Problem 1.** Find the limit

$$\lim_{n\to\infty}\left(\prod_{k=1}^{n}\frac{k}{k+n}\right)^{\left(e^{\frac{1999}{n}}-1\right)}.$$

[10 points]

**Problem 2.** Find all natural numbers $n \geq 1$ such that the implication

$$(11 \mid a^n + b^n) \implies (11 \mid a \wedge 11 \mid b)$$

holds for any two natural numbers $a$ and $b$. [8 points]

**Problem 3.** Let $A_1, \ldots, A_n$ be points of an ellipsoid with center $O$ in $\mathbb{R}^n$ such that $OA_i$, for $i = 1, \ldots, n$, are mutually orthogonal. Prove that the distance of the point $O$ from the hyperplane $A_1A_2 \ldots A_n$ does not depend on the choice of the points $A_1, \ldots, A_n$. [14 points]

**Problem 4.** Show that the following implication holds for any two complex numbers $x$ and $y$: if $x+y$, $x^2+y^2$, $x^3+y^3$, $x^4+y^4 \in \mathbb{Z}$, then $x^n + y^n \in \mathbb{Z}$ for all natural $n$. [8 points]